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NONLINEAR WAVES

Some remarks on the stability and instability properties of solitary waves for the double dispersion equation

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Abstract. In this article we give a review of our recent results on the instability and stability properties of travelling wave solutions of the double dispersion equation $u_{tt}-u_{xx}+au_{xxxx}-bu_{xxtt}=-(|u|^{p-1}u)_{xx}$ for $p>1$, $a\geq b>0$. After a brief reminder of the general class of nonlocal wave equations to which the double dispersion equation belongs, we summarize our findings for both the existence and orbital stability/instability of travelling wave solutions to the general class of nonlocal wave equations. We then state (i) the conditions under which travelling wave solutions of the double dispersion equation are unstable by blow-up and (ii) the conditions under which the travelling waves are orbitally stable. We plot the instability/stability regions in the plane defined by wave velocity and the quotient b/a for various values of p.

Key words: double dispersion equation, Boussinesq equation, solitary waves, instability by blow-up, orbital stability, travelling waves.

1. INTRODUCTION

The present paper provides an overview of results obtained in [1,2] for travelling wave solutions of the double dispersion equation

$$u_{tt}-u_{xx}+au_{xxxx}-bu_{xxtt}=-(|u|^{p-1}u)_{xx}, \tag{1}$$

where $u=u(x,t)$ is a real-valued function, a, b are positive real constants with $a\geq b$, and $p>1$. It also presents the plots of the instability/stability regions of travelling waves for various values of p. We look for solutions to Eq. (1) of the form $u(x,t)=\phi_c(x-ct)$, where $c\in\mathbb{R}$ is the wave velocity and ϕ_c and all its derivatives decay sufficiently rapidly at infinity. We then show that instability by blow-up or orbital stability of travelling waves is observed if certain conditions are met. The simplest definition of orbital stability is that a solution starting close to a travelling wave remains close, at any later time, to some possibly other travelling wave with the same velocity. On the other hand, instability by blow-up means that there are solutions that start arbitrarily close to a travelling wave but blow up in finite time.

The double dispersion equation (1) was introduced as a mathematical model of nonlinear dispersive waves in various contexts (see for instance [3–6] and the references therein). It belongs to the following general class of nonlocal nonlinear wave equations:

$$(I+M)u_{tt}-(I+K)u_{xx}=(g(u))_{xx}, \tag{2}$$

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where I is the identity operator, K and M are linear pseudo-differential operators with smooth symbols $k(\xi)$ and $m(\xi)$, respectively. To see this, notice that when $K = -a\partial_x^2$, $M = -b\partial_x^2$, $g(u) = -|u|^{p-1}u$, Eq. (2) reduces to Eq. (1). Both the Boussinesq equation ($K = -\partial_x^2$, $M = 0$) [7] and the improved Boussinesq equation ($K = 0$, $M = -\partial_x^2$) [8] belong to this class, and they can be considered as the limiting cases of the double dispersion equation. One example for higher-order differential operators is the higher-order improved Boussinesq equation $u_{tt} - u_{xx} - u_{xxtt} + bu_{xxxxtt} = (g(u))_{xx}$ (for which $K = 0$, $M = -\partial_x^2 + b\partial_x^4$) [9]. To consider a truly nonlocal case, taking the operator $(I+M)^{-1}$ as a convolution integral with kernel β and $K = 0$ we get the nonlinear nonlocal wave equation $u_{tt} = [\beta * (u + g(u))]_{xx}$ of nonlocal elasticity [10]. For specific forms of K and M, existence and stability of travelling waves of Eq. (2) were also investigated in [11]. Equation (2) was first introduced in [12] and both global existence and blow-up results for solutions of the initial-value problem with initial data in appropriate function spaces were established. Later, in [13], it was shown that there is a threshold between the global existence and blow-up for solutions of Eq. (2) with power-type nonlinearities.

This review article is organized as follows. Section 2 reviews recent results for travelling wave solutions of Eq. (2). Section 3 reviews recent results for travelling wave solutions of Eq. (1). Section 4 presents the plots of the instability/stability regions of travelling waves for $p = 3$, 5, and 7 as representatives of three possible cases.

2. EXISTENCE AND STABILITY OF TRAVELLING WAVES FOR THE GENERAL CLASS

In this section, we will briefly state the results that were obtained rigorously in [1] for the existence and stability of travelling wave solutions of the general class (2). The basic idea of the existence proof for travelling waves is to show that travelling wave solutions are minimizers of a constrained variational problem. The concentration-compactness lemma of Lions [14,15] is the main tool in obtaining the results, and for a complete understanding of the results we refer the reader to [1]. We now state the three main results of [1].

The first observation in [1] is that travelling wave solutions of the general class (2) with power nonlinearities exist for two different regimes. That is, travelling wave solution $u(x,t) = \phi_c(x - ct)$ of Eq. (2) exists if (i) $c^2 < c_1^2$ and $g(u) = -|u|^{p-1}u$ or, (ii) $c^2 > c_2^2$ and $g(u) = |u|^{p-1}u$. Here the constants c_1 and c_2 are the ellipticity constants satisfying

$$c_1^2(1+\xi^2)^{\rho/2} \leq \frac{1+k(\xi)}{1+m(\xi)} \leq c_2^2(1+\xi^2)^{\rho/2} \tag{3}$$

for all $\xi \in \mathbb{R}$. Thus, in terms of orders of pseudo-differential operators, the two regimes are characterized as follows:

(i) in the first regime, the degree of the symbol $k(\xi)$ is greater than or equal to the degree of the symbol $m(\xi)$ and consequently $\rho \geq 0$, and

(ii) in the second regime, the degree of the symbol $k(\xi)$ is less than or equal to the degree of the symbol $m(\xi)$ and consequently we have $\rho \leq 0$.

The Boussinesq equation ($\rho = 2$) and the improved Boussinesq equation ($\rho = -2$) are the most familiar examples of these two regimes, respectively, with $c_1 = c_2 = 1$. Indeed, the Boussinesq equation has solitary waves for $c^2 < 1$ and $g(u) = -|u|^{p-1}u$, while the improved Boussinesq equation has solitary waves for $c^2 > 1$ and $g(u) = |u|^{p-1}u$. For the double dispersion equation, the degree of the symbol $k(\xi)$ is equal to the degree of the symbol $m(\xi)$ and consequently $\rho = 0$. Thus, as it is expected, the double dispersion equation has solitary wave solutions in both of the two regimes. In particular, it has travelling wave solutions when $c^2 < \min\{1, \frac{a}{b}\}$ and $g(u) = -|u|^{p-1}u$ or when $c^2 > \min\{1, \frac{a}{b}\}$ and $g(u) = |u|^{p-1}u$.

Secondly, orbital stability of travelling waves in the first regime (in which $\rho \geq 0$) was investigated in [1]. It was shown that orbital stability of travelling waves is determined by the convexity of a scalar function of c. In the case of the Boussinesq equation our results reduce to the orbital stability conditions $\frac{p-1}{4} < c^2 < 1$ and

$1 < p < 5$ given in [16]. One result of the analysis in [1] is that, for general operators K and M, travelling wave solutions of Eq. (2) are not orbitally stable when c^2 is sufficiently small. The other result, which was proved using Levine's lemma [17], is that, for suitable initial data close to the standing wave for which $c = 0$, solutions of Eq. (2) blow up in finite time. In the second regime we observe that travelling waves are the saddle points of the constrained variational problem and therefore the method will not work (this is why the corresponding case is called the 'bad case' in [11]).

In [1], as a special case of Eq. (2), the case $K = M$ for which $\rho = 0$ and $c_1 = c_2 = 1$ was also considered. Thus Eq. (2) becomes a class of regularized Klein–Gordon-type equations:

$$u_{tt} - u_{xx} = (I + M)^{-1}(g(u))_{xx}. \tag{4}$$

For the first regime, in which $c^2 < 1$ and $g(u) = -|u|^{p-1}u$, a complete characterization of orbital stability/instability of travelling waves in terms of c was given. Namely, travelling waves are orbitally stable when $c^2 > \frac{p-1}{p+3}$ and they are unstable by blow-up when $c^2 < \frac{p-1}{p+3}$.

3. INSTABILITY/STABILITY OF TRAVELLING WAVES FOR THE DOUBLE DISPERSION EQUATION

In this section, we will briefly review the results that were obtained in [2] for the instability and stability of travelling wave solutions of Eq. (1). In particular, explicit conditions on c, a, b, and p that ensure that travelling waves are orbitally stable or unstable by blow-up are given. We emphasize that both regimes defined in the previous section occur for the double dispersion equation with a general nonlinear function $g(u)$ since $\rho = 0$. However, in this section, we restrict ourselves to the first regime in which $c^2 < 1$ and $g(u) = -|u|^{p-1}u$.

The double dispersion equation admits travelling wave solutions $u(x,t) = \phi_c(x - ct)$ if ϕ_c satisfies the differential equation

$$(a - bc^2)\phi_c'' - (1 - c^2)\phi_c + |\phi_c|^{p-1}\phi_c = 0. \tag{5}$$

Recalling that $a \geq b$, we observe that Eq. (5) has a sech-type solution of the form

$$\phi_c(x) = \left[\frac{1}{2}(p+1)(1-c^2)\right]^{\frac{1}{p-1}} \operatorname{sech}^{\frac{2}{p-1}}\left[\frac{1}{2}(p-1)\left(\frac{1-c^2}{a-bc^2}\right)^{\frac{1}{2}} x\right] \tag{6}$$

if $c^2 < 1$.

The first main result in [2] is that solitary waves (6) of Eq. (1) are unstable by blow-up if $c^2 < c_0^2$ where

$$c_0^2 = \left(\frac{p-1}{p+1}\right)\left[1 + \left(1 - \frac{b(p+3)(p-1)}{a(p+1)^2}\right)^{1/2}\right]^{-1}. \tag{7}$$

That is, for suitable initial data close to the solitary wave (6) with $c^2 < c_0^2$, solutions of the initial-value problem for Eq. (1) blow up in finite time. We underline that, in the limiting case $a = 1$, $b = 0$, our condition $c^2 < c_0^2$ reduces to $c^2 < \frac{p-1}{2(p+1)}$ given in [18] for the Boussinesq equation.

The second main result of [2] is about the orbital stability of solitary wave solutions of Eq. (1). Recall from the discussion in the previous section for Eq. (2) that the orbital stability is based on the convexity of a certain function related to conserved quantities. The explicit forms of the operators K and M for Eq. (1) allow us to compute this scalar function explicitly in the form

$$d(c) = d(0)(1 - c^2)^{\frac{p+3}{2(p-1)}}(1 - \mu c^2)^{\frac{1}{2}}, \tag{8}$$

with $d(0) > 0$ and $0 \leq \mu = \frac{b}{a} \leq 1$. Thus, $d''(c)$ is computed as follows:

$$d''(c) = d(0)(p-1)^{-2}(1-c^2)^{\frac{7-3p}{2(p-1)}}(1-\mu c^2)^{-3/2}(Pc^6 - Qc^4 + Rc^2 - S), \tag{9}$$

where

$$\begin{aligned} P &= 2(p+3)(p+1)\mu^2, \\ Q &= 3(p+3)(p-1)\mu^2 + (3p^2+10p+19)\mu, \\ R &= 2((3p+5)(p-1)\mu + 2(p+3)), \\ S &= (p-1)^2\mu + (p-1)(p+3). \end{aligned}$$

This implies that the sign of $d''(c)$ is determined by the sign of the polynomial

$$G(z,p,\mu) = Pz^3 - Qz^2 + Rz - S \tag{10}$$

with $z = c^2$. Then, the stability regions are the set of all wave velocities c for which $c^2 < 1$ and $G(c^2,p,\mu) > 0$. In [2], using both analytical and numerical techniques, the following observations about the total number of roots in the interval $(0,1)$ were made: (i) For $p < 5$ and $0 \leq \mu \leq 1$, the cubic polynomial $G(z,p,\mu)$ has only one root $z_1(p,\mu)$. (ii) For $p = 5$ and $\frac{1}{3} < \mu < 1$, $G(z,p,\mu)$ has only one root $z_1(p,\mu) = \frac{1}{12\mu}(9-\sqrt{33-24\mu})$. (iii) For $p > 5$ there is a critical value μ_p in $(0,1)$ so that $G(z,p,\mu)$ has no root in $(0,1)$ for $0 \leq \mu < \mu_p$ but it has two roots $z_1(p,\mu)$, $z_2(p,\mu)$ in $(0,1)$ for $\mu_p < \mu < 1$. Based on these observations it was concluded that solitary wave solutions of the double dispersion equation (1) are orbitally stable in each of the following three cases;

$$\text{(A)}\ \ p < 5 \quad \text{and} \quad z_1(p,\mu) < c^2 < 1, \tag{11}$$

$$\text{(B)}\ \ p = 5, \quad \frac{1}{3} < \mu < 1 \quad \text{and} \quad \frac{1}{12\mu}(9-\sqrt{33-24\mu}) < c^2 < 1, \tag{12}$$

$$\text{(C)}\ \ p > 5, \quad \mu_p < \mu < 1 \quad \text{and} \quad z_1(p,\mu) < c^2 < z_2(p,\mu) < 1. \tag{13}$$

Thus three stability regions occur depending on the strength of nonlinearity and the case $p = 5$ is somewhat special.

4. INSTABILITY/STABILITY REGIONS FOR $p = 3,\ 5,$ AND 7

In this section, we plot the instability/stability regions in the $\mu - z$ plane for $p = 3$, $p = 5$, and $p = 7$. In the following figures, the instability curves are defined by $z = c_0^2$, where c_0^2 is given by Eq. (7) whereas the stability curves are defined by the roots of $G(z,p,\mu)$ as in Eqs (11)–(13). For $p = 5$ the stability curve displays the analytical result from Eq. (12). For $p = 3$ and $p = 7$ the stability curves show the roots computed numerically. We remark that the cases $p = 3$ and $p = 7$ are representative cases for $p < 5$ and $p > 5$, respectively. We also remark that the instability curves and the stability curves coincide at the point $(1, c_t^2)$ of the $\mu - z$ plane, where $c_t^2 = \frac{p-1}{p+3}$ is the threshold value between instability by blow-up and orbital stability for the regularized Klein–Gordon type equations. Furthermore, recall that $c_0^2 = \frac{p-1}{2(p+1)}$ at $\mu = 0$.

In Fig. 1, we present the graphs of $z = c_0^2$ and $z = \frac{1}{12\mu}(9-\sqrt{33-24\mu})$ (on $\frac{1}{3} < \mu < 1$ as in Eq. (12)) as a function of μ on the interval $[0,1]$ for $p = 5$. The lower curve is the instability-by-blow-up curve defined by $z = c_0^2$ and the upper curve is the orbital stability curve defined by $z = \frac{1}{12\mu}(9-\sqrt{33-24\mu})$. Thus, the lower shaded region, that is, the area between the curve $z = c_0^2$ and the line $z = 0$, corresponds to the region of instability by blow-up for $p = 5$. Similarly, the upper shaded region, that is, the area between the curve $z = \frac{1}{12\mu}(9-\sqrt{33-24\mu})$ and the line $z = 1$, corresponds to the region of orbital stability for $p = 5$. It is

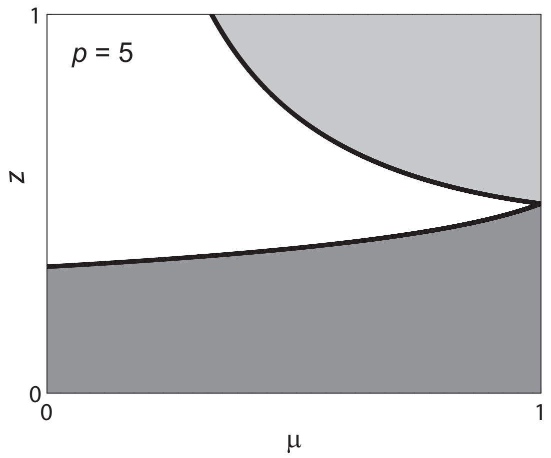

Fig. 1. Variation of $z(=c^2)$ with μ on the interval $[0,1]$ for $p=5$ when $z=c_0^2$ (lower curve) and when $z=z_1(p,\mu)$ (upper curve). The lower and upper shaded regions represent the region of instability by blow-up and the region of orbital stability, respectively.

interesting to note that it is unclear what happens in the interval $c_0^2 < c^2 < \frac{1}{12\mu}(9-\sqrt{33-24\mu})$; this remains as an open question. We also observe that the size of the gap region between the shaded regions decreases as μ increases. In other words, for $p=5$, the stability interval of c^2 gets larger as μ increases.

In Fig. 2, we present the graphs of $z=c_0^2$ and $z=z_1(p,\mu)$ (the root $z_1(p,\mu)$ appears in Eq. (11)) as a function of μ on the interval $[0,1]$ for $p=3$. The lower curve is again the instability-by-blow-up curve defined by $z=c_0^2$ and the upper curve is again the orbital stability curve defined by $z=z_1(p,\mu)$. Thus, the lower shaded region, that is, the area between the curve $z=c_0^2$ and the line $z=0$, corresponds to the region of instability by blow-up for $p=3$. Similarly, the upper shaded region, that is, the area between the curve $z=z_1(p,\mu)$ and the line $z=1$, corresponds to the region of orbital stability for $p=3$. All the comments made for Fig. 1 are valid for Fig. 2 as well.

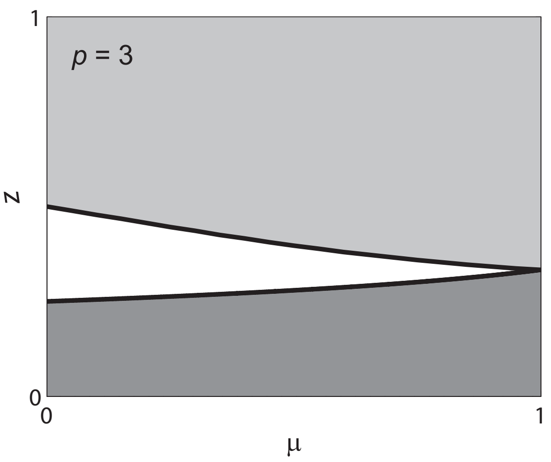

Fig. 2. Variation of $z(=c^2)$ with μ on the interval $[0,1]$ for $p=3$ when $z=c_0^2$ (lower curve) and when $z=z_1(p,\mu)$ (upper curve). The lower and upper shaded regions represent the region of instability by blow-up and the region of orbital stability, respectively.

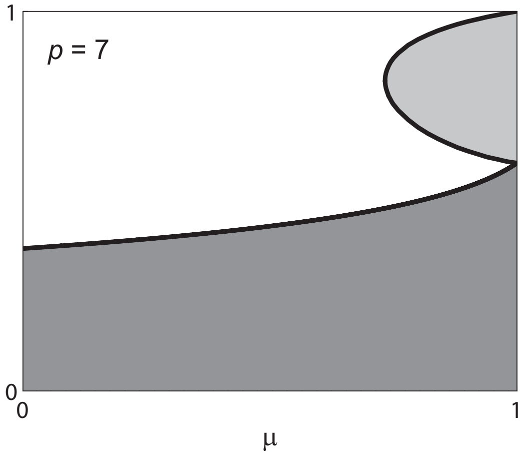

Fig. 3. Variation of $z(=c^2)$ with μ on the interval $[0,1]$ for $p=7$ when $z=c_0^2$ (lower curve), when $z=z_1(p,\mu)$ (middle curve), and $z=z_2(p,\mu)$ (upper curve). The lower and upper shaded regions represent the region of instability by blow-up and the region of orbital stability, respectively.

In Fig. 3, we present the graphs of $z=c_0^2$, $z=z_1(p,\mu)$, and $z=z_2(p,\mu)$ (the roots $z_1(p,\mu)$ and $z_2(p,\mu)$ appear in Eq. (13)) as a function of μ on the interval $[0,1]$ for $p=7$. The lower curve is again the instability-by-blow-up curve defined by $z=c_0^2$ and the upper curves are the orbital stability curves defined by $z=z_1(p,\mu)$ and $z=z_2(p,\mu)$. Thus, the lower shaded region, that is, the area between the curve $z=c_0^2$ and the line $z=0$, corresponds to the region of instability by blow-up for $p=7$. Similarly, the upper shaded region, that is, the area between the curve $z=z_1(p,\mu)$ and the curve $z=z_2(p,\mu)$, corresponds to the region of orbital stability for $p=7$. We note that a bifurcation occurs at the critical value $\mu_p=0.7333$, which separates the case of $\mu<0.7333$ in which $G(z,p,\mu)$ has no real root z in the interval $(0,1)$ from the case of $\mu>0.7333$ in which $G(z,p,\mu)$ has two real roots z_1 and z_2 in $(0,1)$. All the comments made for Fig. 1 are also valid for Fig. 3. It is interesting to note that an additional gap region defined by $z=z_2(p,\mu)<c^2<1$ appears for $p=7$.

In Figs 2 and 3, which are representatives of the cases $p<5$ and $p>5$, respectively, one notices the sudden change of the stability region occurring at $p=5$ (see Fig. 1). At first sight this behaviour seems to contradict continuity with respect to the parameter p. Nevertheless, a closer observation shows that when $p=5$, the polynomial $G(z,5,\mu)$ has a root at $z=1$. This root, which is not seen in Fig. 1, indeed explains the bifurcation at $p=5$ occurring at $z=1$.

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Mõned märkused üksiklainete stabiilsuse ja ebastabiilsuse kohta topeltdispersioonivõrrandi korral

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Käesolevas artiklis anname ülevaate oma hiljutistest tulemustest üksiklainete stabiilsuse ja ebastabiilsuse kohta topeltdispersioonivõrrandi korral. Me näitame, millisel juhul eksisteerivad lahendid leviva laine kujul, millisel juhul on need lahendid orbitaalselt stabiilsed ja millisel juhul ebastabiilsed.