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Research paper

Captive jump processes for bounded random systems with discontinuous dynamics

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ABSTRACT

Stochastic captive jump processes are explicitly constructed in continuous time, whose non-linear dynamics are strictly confined by bounded domains that can be time-dependent. By introducing non-anticipative path-dependency, the framework offers the possibility of generating multiple inner tunnels within a master domain, such that a captive jump process is allowed to proceed either within a single inner tunnel or jump in between tunnels without ever penetrating the outermost shell. If a captive jump process is a continuous martingale or a pure-jump process, the uppermost confining boundary is non-decreasing, and the lowermost confining boundary is non-increasing. Under certain conditions, it can be shown that captive jump processes are invariant under monotonic transformations, enabling one to construct and study systems of increasing complexity using simpler building blocks. Amongst many applications, captive jump processes may be considered to model phenomena such as electrons transitioning from one orbit (valence shell) to another, quantum tunnelling where stochastic wave-functions can "penetrate" inner boundaries (i.e., walls) of potential energy, non-linear dynamical systems involving multiple attractors, and sticky concentration behaviour of pathogens in epidemics. We provide concrete, worked-out examples, and numerical simulations for the dynamics of captive jump processes within different geometries as demonstrations.

1. Introduction

In this paper we aim at establishing a mathematical framework for what we refer to as *captive jump processes* (CJPs). These are stochastic processes restricted within time-dependent, confined path spaces. CJPs extensively generalize the captive diffusion processes in [1] by including a class of pure-jump processes, jump-diffusions and path-dependent processes that can evolve across disparate segments of a complex geometry. This in turn paves the way to a wide array of potential applications in nonlinear systems that consist of a single master domain and/or multiple sub-domains (e.g., inner tunnels or corridors) within that master domain. It is this master domain (with outermost boundaries) from whence the CJP can never escape, even though the same CJP is free (under certain conditions) to move through, and jump across, multiple fractures that fall in between.

In many physical and social systems, one encounters random processes that are *restricted* in their dynamics, where the stochastic phenomenon stays within a given topological subspace. As a most closely related stream of literature (which excludes jumps), we refer to [2–4] for examples in quadratic optimization for generating risk-bounded, efficient frontiers, fully stochastic order-preserving

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systems, and randomly tunnelled particle systems. We shall also emphasize several other constructions, such as Skorokhod-type stochastic differential equations [5–8], diffusion processes on submanifolds [9–11], reflected diffusions [12–15], Brownian excursions [16–18], non-colliding diffusions [19–22], and Bessel processes [23–26].

The aforementioned stochastic processes share the common property of having continuous paths. However, there are many observed nonlinear systems which display discontinuities (at random times), possibly due to shocks producing substantial impact on the evolution of the system. Here we mention transitions of quantum systems across excitation levels (e.g., electrons jumping to a different energy level, sudden price reactions to news in financial markets, policy interventions regulating traffic in networks, and circuit-breakers). Thus, studying processes that embed pure-jump or jump–diffusion dynamics justifies a great deal of interest in numerous applications and mathematical modelling purposes—if not just for their own sake.

Our approach considerably enlarges the variety of dynamics that captive processes can display, and addresses new fields of applications in natural, life and social sciences that demand an accurate representation of discontinuities in multi-regime environments, which would not be captured in the purely continuous setting, see [1,3,4]. The proposed generalized family of processes retains the core features of the continuous case in [1] (that relies on the particular ordering of right-derivatives at boundaries), and additionally permits random jumps that occur discretely and endogenously within the strict boundaries of a master domain (almost-surely or path-by-path). This extension also brings the flexibility of creating more advanced topological designs compared to [4] when studying path-dependent jumps that move across internal corridors (e.g., self-contained subspaces or regimes) in a given master system. We emphasize that adding jumps into the framework is hence non-trivial. Since we can consider highly complex geometries within our framework, the architecture of discontinuities requires an intricate mathematical design which we achieve through a path-dependent monitoring process that dynamically records the tunnel entry points and random jump times. CJPs can display numerous properties (pure-jump, continuous or jump-continuous) over non-overlapping time intervals throughout their lifetime, and are controlled by coefficients of stochastic differential equations (SDEs) that satisfy certain regularity conditions with respect to exogenous boundaries. Such processes are now governed by jump-times and the corresponding spatial positions—all being measurable at any given time. As such, we shall see that this flexibility is desirable, and offers fundamental advancement in several directions in the developing literature on domain-bounded processes.

There are, of course, other ways to generate bounded stochastic processes; one can apply bounded functions to a stochastic process, or directly truncate its law via transforming its probability distribution. Such approaches, however, might not always capture the complex properties of an ecosystem, in which boundedness may not be the only characteristic feature that underlies the evolution of the processes involved. For example, an ecosystem may consist of multiple subspaces that host time-inhomogeneous topological fragmentation, such that the underlying stochastic processes are expected to adopt this geometry in an order-preserving and path-dependent manner. In this sense, these processes may proceed across non-overlapping regime changes, while demonstrating "absorption" or "reflection" behaviour at the boundaries. Such an environment could (i) require the dynamics to be explicitly dependent on confining boundaries and the induced geometry, and (ii) represent a dynamic interaction between all agents in a multivariate setting. Not only do we propose a mathematical recipe that unifies all these aspects in a single framework, but also aim at developing this framework in a flexible and tractable fashion through parametric SDEs that admit numerical calibration and simulation for practical applications. We shall see that conducting numerical simulations requires careful examination given the extended flexibility of the possible systems our framework can model. In order to give evidence for the applicability of our construction, we numerically simulate various path spaces by increasing their degree of complexity (in terms of the level of fragmentation and non-linearity of boundaries), and discuss the performance of the obtained numerical models in light of potential boundary breaches.

The structure of this paper is as follows. Section 2 introduces captive jump–diffusion processes and studies some of their mathematical properties. In Section 3, we apply the framework to various physical phenomena along with numerical simulations and their sensitivity analysis. Additional remarks and conclusions are found in Sections 4 and 5 respectively.

2. Construction and properties of captive jump processes

We consider a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\leq\infty}, \mathbb{P})$, $\mathcal{F}_\infty = \mathcal{F}$, where all filtrations are right-continuous and complete. We work over a compact time interval $\mathbb{T} = [0, T]$ with some fixed horizon $T < \infty$. Following the presentation of random processes as random functions in [27], Section 5.2, we introduce for each $\omega \in \Omega$ the function $t \mapsto X(t, \omega) \in \mathbb{R}$, for $t \in \mathbb{T}$. This function is commonly referred to as an ω-trajectory or ω-sample path of the stochastic process $(X_t)_{t\in\mathbb{T}}$. If for all $\omega \in \Omega$ the ω-sample path is càdlàg (right-continuous with left limits), then $(X_t)_{t\in\mathbb{T}}$ is said to be a càdlàg stochastic process. If for all $\omega \in \Omega$, except for a $\mathbb{P}$-nullset, the ω-sample path is càdlàg , then $(X_t)_{t\in\mathbb{T}}$ is said to be a $\mathbb{P}$-almost sure càdlàg stochastic process. Other ω-sample path properties are defined in a similar way, see [27]. By $(\mathcal{F}^X_t)_{t\in\mathbb{T}}$ we denote the natural filtration of $(X_t)_{t\in\mathbb{T}}$ where $\mathcal{F}^X_t \subset \mathcal{F}_t$ is a sub-algebra for any $t \in \mathbb{T}$. We choose to introduce the process $(X_t)_{t\in\mathbb{T}}$ as a random function because the ensuing construction of the captive jump processes will rely on constraining the path space of the process's ω-trajectories by employing time-dependent boundary functions. We shall begin with captive jump processes that evolve within two master boundaries. Thereafter, we generalize the setting so to allow for internal corridors to emerge.

2.1. Captive jump–diffusion processes

We introduce the space $\mathcal{M}(\mathbb{R})$ of $\mathbb{P}$-almost sure continuous $\{(\mathcal{F}_t), \mathbb{P}\}$-martingales taking values in $\mathbb{R}$. Let $\mathcal{J}(\mathbb{R})$ be the space of $\mathbb{P}$-almost sure càdlàg jump processes $(J_t)_{t\in\mathbb{T}}$ with finite activity (i.e., finite number of jumps in a finite time interval) and jump-size one. We write $\Delta J_t = J_t - J_{t-}$ for $t > t-$, for all $t \in \mathbb{T}$. So, if $\Delta J_t = 0$, then the process (J_t) is continuous at t. If $\Delta J_t = 1$, then there is discontinuity at t. For any $f : \mathbb{T} \to \mathbb{R}$ we shall use f_t and $f(t)$ interchangeably, depending on notational convenience. The following function space models the boundaries that determine the restricted domains.

Definition 2.1. Let $\mathcal{G}(\mathbb{R})$ be the space of continuous and deterministic functions taking values in $\mathbb{R}$, where for any $g \in \mathcal{G}(\mathbb{R})$, $g : \mathbb{T} \to \mathbb{R}$ is a locally bounded map with a locally bounded right-derivative $\mathrm{d}g_+(t)/\mathrm{d}t$.

We now introduce a family of processes that generalizes [1] and forms the main focus of this paper. Thereafter, we study the main properties of this class of processes and provide some examples.

Definition 2.2. Let $g^l \in \mathcal{G}(\mathbb{R})$ and $g^u \in \mathcal{G}(\mathbb{R})$ such that $g^l(t) < g^u(t)$ for all $t \in \mathbb{T}$. Then, a *captive jump–diffusion process* $(X_t)_{t\in\mathbb{T}}$ is a solution to the SDE

$$X_t = x_0 + \int_0^t \mu\left(s, X_s; g^l(s), g^u(s)\right) \mathrm{d}s + \int_0^t \sigma\left(s, X_s; g^l(s), g^u(s)\right) \mathrm{d}M_s + \sum_{0\le s\le t} \gamma\left(s-, X_{s-}; g^l(s-), g^u(s-)\right) \Delta J_s, \tag{1}$$

where $X_0 = x_0 \in [g^l(0), g^u(0)]$. The maps μ and σ are continuous (possibly except at points where $\Delta J \neq 0$ with bounded jumps), and γ is a locally bounded càdlàg map such that

1. $\mu\left(t, g^l(t); g^l(t), g^u(t)\right) \ge \mathrm{d}g^l_+(t)/\mathrm{d}t$ and $\mu\left(t, g^u(t); g^l(t), g^u(t)\right) \le \mathrm{d}g^u_+(t)/\mathrm{d}t$,
2. $\sigma\left(t, g^l(t); g^l(t), g^u(t)\right) = 0$ and $\sigma\left(t, g^u(t); g^l(t), g^u(t)\right) = 0$,
   where 1. and 2. above holds for any $t \in \mathbb{T}$ where $X_t = g^l(t)$ or $X_t = g^u(t)$, and
3. $g^l(t-) - X_{t-} \le \gamma\left(t-, X_{t-}; g^l(t-), g^u(t-)\right) \le g^u(t-) - X_{t-}$,

holds for all $t \in \mathbb{T}$, $\mathbb{P}$-a.s., given that $(M_t)_{t\in\mathbb{T}} \in \mathcal{M}(\mathbb{R})$ and $(J_t)_{t\in\mathbb{T}} \in \mathcal{J}(\mathbb{R})$ are mutually independent.

To maintain a flexible level of generality, we *define* captive jump–diffusion processes as solutions governed by Eq. (1) without imposing specific conditions on the coefficients of the SDE for existence and uniqueness. In order to guarantee a unique (strong) solution, one can further require the coefficients to satisfy Lipschitz continuity and linear growth, but these are not necessary conditions.

Proposition 2.3. *For any $t \in \mathbb{T}$, $g^l(t) \le X_t \le g^u(t)$ holds $\mathbb{P}$-almost-surely.*

Proof. The continuous case is recovered if $\gamma\left(t, X_t; g^l(t), g^u(t)\right) = 0$ for all $t \in \mathbb{T}$, which is proven in [1]. We have $\Delta X_t \neq 0$ only when $\Delta J_t \neq 0$, where the magnitude of ΔX_t is conditionally restricted for all $t \in \mathbb{T}$ by the third property in Definition 2.2. Hence, jumps can at most take $(X_t)_{t\in\mathbb{T}}$ onto a boundary, where the condition on $\sigma(\cdot)$ ensures that X_t at the boundary ($X_t = g^l(t)$) is right-differentiable (given that $\Delta J_t = 0$ at that $t \in \mathbb{T}$), so that we have

$$\mu\left(t, g^l(t); g^l(t), g^u(t)\right) = \lim_{\epsilon\to 0^+} \frac{X_{t+\epsilon} - g^l(t)}{\epsilon} \ge \lim_{\epsilon\to 0^+} \frac{g^l(t+\epsilon) - g^l(t)}{\epsilon}. \tag{2}$$

Since $(J_t)_{t\in\mathbb{T}}$ has finite activity, if there is no jump at the boundary-hitting time $t \in \mathbb{T}$ where $X_t = g^l(t)$, there exists an ϵ-band for which Eq. (2) holds. If there is jump at the boundary, then the process jumps back into the domain (or remains on the boundary) $\mathbb{P}$-a.s. by virtue of Property 3 in Definition 2.2. Hence, these imply $X_t \ge g^l(t)$ for any $t \in \mathbb{T}$, $\mathbb{P}$-a.s. Similarly, using the condition on $\sigma(\cdot)$, the opposite side follows the same logic, where at $X_t = g^u(t)$, we have

$$\mu\left(t, g^u(t); g^l(t), g^u(t)\right) = \lim_{\epsilon\to 0^+} \frac{X_{t+\epsilon} - g^u(t)}{\epsilon} \le \lim_{\epsilon\to 0^+} \frac{g^u(t+\epsilon) - g^u(t)}{\epsilon},$$

if there is no jump during the ϵ-band over which the limit above is taken. If there is a jump at the boundary, Property 3 in Definition 2.2 dictates $(X_t)_{t\in\mathbb{T}}$ to jump back into the domain. Hence, $X_t \le g^u(t)$ must hold for any $t \in \mathbb{T}$, $\mathbb{P}$-a.s. □

Remark 2.4. The reason for calling these processes "captive" stems from Proposition 2.3: The process cannot break free from the restricted domain controlled by $g^l(t)_{t\in\mathbb{T}}$ and $g^u(t)_{t\in\mathbb{T}}$.

At this point, we shall mention the literature on *comparison theorems*. These theorems give necessary and sufficient conditions for pairs of SDEs that maintain an initial order, see [28–31]. Although the conditions on drift and diffusion coefficients discussed in this paper (and in [1,3]) show some similarities with the conditions in the comparison theorems, we shall highlight several fundamental differences between the two approaches. First, captive jump processes are driven by multivariate drift and diffusion coefficients which are in one-to-one relations with the boundaries, whereby comparison theorems work with one-dimensional functions on the underlying processes without any reference to boundary paths. Second, comparison theorems require the drift coefficients to abide to a pairwise-order at *every* point in space–time and require the diffusion coefficients to be the same function at *every* point in space–time (to ensure order-preservation due to the one-dimensional nature of the employed functions), whereby our framework does *not* expect the conditions imposed on the coefficients to be satisfied everywhere. Instead, conditions are to be satisfied dynamically only at collision events in space–time on multivariate surfaces. The latter feature grants to our framework the flexibility to choose from a considerably large family of coefficient functions. Third, captive jump processes form a class of dynamically (and randomly) *degenerate* SDEs which are, in general, *only* right-differentiable *at* boundaries (and in general non-differentiable anywhere else in their domain); a feature that does not appear in the comparison theorems. Finally, comparison

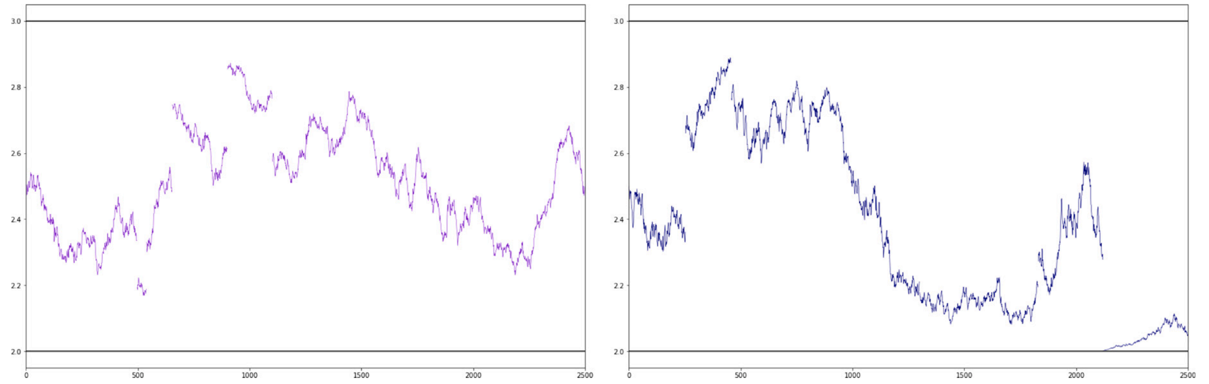

Fig. 1. Here, $L(t)_{t\in\mathbb{T}} = 2$ and $U(t)_{t\in\mathbb{T}} = 3$, and $\beta(t)_{t\in\mathbb{T}} = 2.5$, $\alpha = 1.0$.

theorems ask for a *single* stochastic driver associated to every diffusion coefficient in the system, disallowing the manifestation of multiple cases of idiosyncratic randomness associated individually to every diffusion coefficient (as shown in [3]). It is due to these reasons that comparison theorems and captive (jump) processes (see also [1,3]) diverge, aside from the very specific case, where every diffusion coefficient function is fixed to constant zero and every drift coefficient satisfies a fixed order at *every* point in space–time, while diffusion and drift functions are one-dimensional. The very limiting special case boils down to essentially a purely deterministic system that loses all possible stochastic dynamics. Accordingly, our framework deviates considerably, leading to different proofs, results and extensions, when compared to comparison theorems.

Remark 2.5. Since in general neither $(M_t)_{t\in\mathbb{T}}$ nor $(J_t)_{t\in\mathbb{T}}$ are necessarily Markovian, captive jump–diffusion processes are not necessarily Markov processes, either. The term "diffusion" in this paper is to highlight continuity of paths rather than Markovianity. When $(X_t)_{t\in\mathbb{T}}$ is Markovian, we replace $(M_t)_{t\in\mathbb{T}}$ in Eq. (1) with an $\{(\mathcal{F}_t),\mathbb{P}\}$-Brownian motion $(W_t)_{t\in\mathbb{T}}$ for a canonical representation, and choose for $(J_t)_{t\in\mathbb{T}}$ an independent Markov jump process.

Remark 2.6. One can further augment $\mathcal{M}(\mathbb{R})$ from the space of continuous martingales to that of continuous semimartingales. That is, Proposition 2.3 would still hold if $(M_t)_{t\in\mathbb{T}}$ is a continuous semimartingale. In this paper, we keep $(M_t)_{t\in\mathbb{T}}$ a continuous martingale since $(M_t)_{t\in\mathbb{T}}$ will be chosen as a Brownian motion in all our examples.

For demonstration purposes, we shall next provide an example for a Markovian captive jump–diffusion process.

Example 2.7. Let $(J_t)_{t\in\mathbb{T}}$ be a Poisson process. The following process is a mean-reverting captive jump–diffusion with reflective boundaries:

$$X_t = x_0 + \int_0^t \kappa(s)(\beta(s) - X_s)\mathrm{d}s + \int_0^t \alpha(s)\left(X_s - L(s)\right)\left(U(s) - X_s\right)\mathrm{d}W_s + \sum_{0\le s\le t} \theta_{s-} \min\left(X_{s-} - L(s-), U(s-) - X_{s-}\right)\Delta J_s, \tag{3}$$

for $x_0 \in [L_0, U_0]$, where $(\alpha(t))_{t\in\mathbb{T}}$ and $0 < (\kappa(t))_{t\in\mathbb{T}} < \infty$ are adapted continuous maps, and the following relationship holds:

$$L(t) < \beta(t) < U(t),$$

for all $t \in \mathbb{T}$, so that

$$\kappa(t)(\beta(t) - L(t)) > \mathrm{d}L_+(t)/\mathrm{d}t$$
$$\kappa(t)(\beta(t) - U(t)) < \mathrm{d}U_+(t)/\mathrm{d}t,$$

for all $t \in \mathbb{T}$. Also, $(\theta_t)_{t\in\mathbb{T}}$ is a (possibly stochastic) càdlàg map where $\theta_t \in [-1,1]/\{0\}$, for all $t \in \mathbb{T}$. In Fig. 1, we plot sample paths generated by a simplified version of the SDE (3), where we consider constant boundaries $L(t)_{t\in\mathbb{T}} = L$ and $U(t)_{t\in\mathbb{T}} = U$, and constant $\beta(t)_{t\in\mathbb{T}} = \beta$.

Remark 2.8. We note that all the coefficients in (3) satisfy local Lipschitz continuity. It follows that the SDE is well-posed and a solution exists, which is bounded according to Proposition 2.3.

Since sums of semimartingales are semimartingales, if $(J_t)_{t\in\mathbb{T}}$ is a $\{(\mathcal{F}_t),\mathbb{P}\}$- semimartingale, then $(X_t)_{t\in\mathbb{T}}$ is a $\{(\mathcal{F}_t),\mathbb{P}\}$-semimartingale given that μ has locally bounded variation. For our next result, we define the time-segments $\mathcal{T}^{(l)} \subseteq \mathbb{T}$ and $\mathcal{T}^{(u)} \subseteq \mathbb{T}$ where $(X_t)_{t\in\mathbb{T}}$ may hit the boundaries. We have,

$$\mathcal{T}^{(l)} = \{t : \mathbb{P}(X_t \in \delta g^l(t)) > 0;\ \text{for } t \in \mathbb{T}\},$$

$$\mathcal{T}^{(u)} = \{t : \mathbb{P}(X_t \in \delta g^u(t)) > 0; \;\text{ for } t \in \mathbb{T}\},$$

where $\delta g^l(t)$ and $\delta g^u(t)$ are the spatial differentials for a fixed time $t \in \mathbb{T}$. That is, if $g(t) : t \mapsto x$ then $\delta g(t) = \mathrm{d}x$.

Proposition 2.9. *Let $g^l, g^u \in \mathcal{G}$. Then $(g^l(t))_{t\in\mathcal{T}^{(l)}}$ must be non-increasing and $(g^u(t))_{t\in\mathcal{T}^{(u)}}$ must be non-decreasing if either of the following holds:*

1. *$(X_t)_{t\in\mathbb{T}}$ is a continuous $\{(\mathcal{F}_t), \mathbb{P}\}$-martingale,*
2. *$(X_t)_{t\in\mathbb{T}}$ is a pure-jump process.*

Proof. Since $g^l, g^u \in \mathcal{G}$, we have $\Delta g^l(t) = \Delta g^u(t) = 0$ for any $t \in \mathbb{T}$. For the first part, if $(X_t)_{t\in\mathbb{T}}$ is a continuous $\{(\mathcal{F}_t), \mathbb{P}\}$-martingale then $\mu = \gamma = 0$. Since

$$\mu\left(t, g^l(t); g^l(t), g^u(t)\right) = 0 \geq \mathrm{d}g^l_+(t)/\mathrm{d}t$$

must hold for all $t \in \mathcal{T}^{(l)}$ and

$$\mu\left(t, g^u(t); g^l(t), g^u(t)\right) = 0 \leq \mathrm{d}g^u_+(t)/\mathrm{d}t$$

must hold for all $t \in \mathcal{T}^{(u)}$, it follows that $(g^l(t))_{t\in\mathcal{T}^{(l)}}$ must be non-increasing and $(g^u(t))_{t\in\mathcal{T}^{(u)}}$ must be non-decreasing. For the second part, if $(X_t)_{t\in\mathbb{T}}$ is a pure-jump process, then $\mu = \sigma = 0$. Since the jump-times of $(J_t)_{t\in\mathbb{T}}$ neither depend on $(g^l(t))_{t\in\mathbb{T}}$ nor on $(g^u(t))_{t\in\mathbb{T}}$, and since γ cannot act on $(X_t)_{t\in\mathbb{T}}$ over time intervals where $(J_t)_{t\in\mathbb{T}}$ does not jump, hence where $(X_t)_{t\in\mathbb{T}}$ is constant, we have the following:

$$\begin{aligned}
&\mathbb{P}(X_t > g^u(t)) > 0 \quad \text{ if } \mathrm{d}g^u_+(t)/\mathrm{d}t < 0,\\
&\mathbb{P}(X_t < g^l(t)) > 0 \quad \text{ if } \mathrm{d}g^l_+(t)/\mathrm{d}t > 0,
\end{aligned}$$

for $t \in \mathbb{T}$. However, the captivity property is preserved if $\mathrm{d}g^u_+(t)/\mathrm{d}t \geq 0$ and $\mathrm{d}g^l_+(t)/\mathrm{d}t \leq 0$, since the boundaries cannot evolve in a way that cut the paths of $(X_t)_{t\in\mathbb{T}}$ over time intervals where no jumps occur, and hence over periods during which $(X_t)_{t\in\mathbb{T}}$ stays constant. □

Proposition 2.10. *Let $(g^l(t))_{t\in\mathbb{T}}$ and $(g^u(t))_{t\in\mathbb{T}}$ be constant and $\mathbb{E}^{\mathbb{P}}[\Delta J_t \mid \mathcal{F}_{t-}] > 0$ for all $t \in \mathbb{T}$. Then the boundaries must be absorbing if*

1. *$(X_t)_{t\in\mathbb{T}}$ is a continuous $\{(\mathcal{F}_t), \mathbb{P}\}$-martingale,*
2. *$(X_t)_{t\in\mathbb{T}}$ is a pure-jump $\{(\mathcal{F}_t), \mathbb{P}\}$-martingale.*

Proof. Denote $g^l(t) = L_t$, $g^u(t) = U_t$ for all $t \in \mathbb{T}$ and let

$$\tau = \inf(t \geq 0 : X_t = L_t = L \cup X_t = U_t = U)$$

be the first-hitting time to either of the (constant) boundaries where $\inf \emptyset = \infty$.

Next, define $(Y_t)_{t\in\mathbb{T}}$ such that

$$Y_t = X_{t\wedge\tau} \quad \text{ for all } t \in \mathbb{T}.$$

Since $(L_t)_{t\in\mathbb{T}}$ and $(U_t)_{t\in\mathbb{T}}$ are constant, we have $\mathrm{d}L_+(t)/\mathrm{d}t = \mathrm{d}U_+(t)/\mathrm{d}t = 0$. If $(X_t)_{t\in\mathbb{T}}$ is an $\{(\mathcal{F}_t), \mathbb{P}\}$-martingale, then we have $\mathbb{E}^{\mathbb{P}}[|X_t|] < \infty$ and the following:

$$\begin{aligned}
\mathbb{E}^{\mathbb{P}}[X_t \mid \mathcal{F}_u] = x_0 &+ \int_0^u \mu\left(s, X_s; L_s, U_s\right) \mathrm{d}s + \int_0^u \sigma\left(s, X_s; L_s, U_s\right) \mathrm{d}M_s + \sum_{0\leq s\leq u} \gamma\left(s-, X_{s-}; L_{s-}, U_{s-}\right) \Delta J_s \\
&+ \mathbb{E}^{\mathbb{P}}\left[\int_u^t \mu\left(s, X_s; L_s, U_s\right) \mid \mathcal{F}_u\right] \mathrm{d}s + \mathbb{E}^{\mathbb{P}}\left[\int_u^t \sigma\left(s, X_s; L_s, U_s\right) \mathrm{d}M_s \mid \mathcal{F}_u\right] \\
&+ \mathbb{E}^{\mathbb{P}}\left[\sum_{u<s\leq t} \gamma\left(s-, X_{s-}; L_{s-}, U_{s-}\right) \Delta J_s \mid \mathcal{F}_u\right] \\
= X_u &+ \int_u^t \mathbb{E}^{\mathbb{P}}[\mu\left(s, X_s; L_s, U_s\right) \mid \mathcal{F}_u]\mathrm{d}s + \sum_{u<s\leq t} \mathbb{E}^{\mathbb{P}}[\gamma\left(s-, X_{s-}; L_{s-}, U_{s-}\right) \Delta J_s \mid \mathcal{F}_u] \\
= X_u&,
\end{aligned} \tag{4}$$

for all $u \leq t \in \mathbb{T}$. If $(X_t)_{t\in\mathbb{T}}$ is continuous, then $\gamma = 0$ and $\mu = 0$ must hold, and since $\sigma(\tau) = 0$ by Definition 2.2, we must have $(X_t)_{t\in\mathbb{T}} = (Y_t)_{t\in\mathbb{T}}$. If $(X_t)_{t\in\mathbb{T}}$ is a pure-jump process, then $\mu = \sigma = 0$, and by using the independence of $(J_t)_{t\in\mathbb{T}}$, the following must hold:

$$\sum_{u<s\leq t} \mathbb{E}^{\mathbb{P}}[\gamma\left(s-, X_{s-}; L_{s-}, U_{s-}\right) \Delta J_s \mid \mathcal{F}_u] = \sum_{u<s\leq t} \mathbb{E}^{\mathbb{P}}[\gamma\left(s-, X_{s-}; L_{s-}, U_{s-}\right) \mid \mathcal{F}_u] \mathbb{E}^{\mathbb{P}}[\Delta J_s \mid \mathcal{F}_u] = 0.$$

Since this holds for every $u < s \leq t \in \mathbb{T}$,

$$\mathbb{E}^{\mathbb{P}}[\Delta J_s \mid \mathcal{F}_u] > 0 \;\Rightarrow\; \mathbb{E}^{\mathbb{P}}[\gamma\left(s-, X_{s-}; L_{s-}, U_{s-}\right) \mid \mathcal{F}_u] = 0$$

must hold for every $u < s \le t \in \mathbb{T}$. If $\tau \in \mathbb{T}$, that is, the captive process jumped onto a boundary during $\mathbb{T}$, and if $\rho \in \mathbb{T}$ is any time strictly *after* τ at which $(J_t)_{t\in\mathbb{T}}$ jumps (this always holds due to the finite-activity property), we have:

$$\mathbb{E}^{\mathbb{P}}[\gamma\left(\rho-, X_{\rho-}; L_{\rho-}, U_{\rho-}\right) \mid \mathcal{F}_\tau] = 0 \Rightarrow \begin{cases} \mathbb{E}^{\mathbb{P}}[\gamma\left(\rho-, L_{\rho-}; L_{\rho-}, U_{\rho-}\right) \mid \mathcal{F}_\tau] = 0 & \text{if } X_\tau = L_\tau, \\ \mathbb{E}^{\mathbb{P}}[\gamma\left(\rho-, U_{\rho-}; L_{\rho-}, U_{\rho-}\right) \mid \mathcal{F}_\tau] = 0 & \text{if } X_\tau = U_\tau \end{cases} \tag{5}$$

for $\tau < \rho \in \mathbb{T}$. Using Property 3 in Definition 2.2, we have one of the following cases:

$$\gamma\left(\rho-, L_{\rho-}; L_{\rho-}, U_{\rho-}\right) \ge 0,$$
$$\gamma\left(\rho-, U_{\rho-}; L_{\rho-}, U_{\rho-}\right) \le 0.$$

In any of these two cases, Eq. (5) holds only if $\gamma\left(\rho-, X_{\rho-}; L_{\rho-}, U_{\rho-}\right) = 0$ for any $\rho > \tau$. Thus, $(X_t)_{t\in\mathbb{T}} = (Y_t)_{t\in\mathbb{T}}$ must hold and the statement follows. □

We now ask: Is there a family of transformations under which a captive jump–diffusion is mapped to another captive jump–diffusion? The answer is yes. This answer carries value since it allows one to construct more advanced captive jump processes starting from simpler models. In order to formalize this answer, first let $C_b^2(\mathbb{R})$ be the subspace of continuous locally bounded functions that are twice-differentiable with continuous locally bounded derivatives. From this point onwards, $(X_t)_{t\in\mathbb{T}}$ is a Markovian captive jump–diffusion process as described in Remark 2.5. We shall recall that the jump-times of $(J_t)_{t\in\mathbb{T}}$ are mutually independent of $(X_t)_{t\in\mathbb{T}}$.

Proposition 2.11. *Let $g^l, g^u \in \mathcal{G}(\mathbb{R})$, μ and σ be continuous, $f \in C_b^2(\mathbb{R})$ and $Y_t = f(X_t)$ for all $t \in \mathbb{T}$. If f is strictly monotonic over the domain of $(X_t)_{t\in\mathbb{T}}$, then $(Y_t)_{t\in\mathbb{T}}$ is a captive jump process.*

Proof. Denote $g^l(t) = L_t$, $g^u(t) = U_t$ for all $t \in \mathbb{T}$. Since f is monotonic, either $f(L_t) \le Y_t \le f(U_t)$ if f is increasing, or $f(U_t) \le Y_t \le f(L_t)$ if f is decreasing, for all $t \in \mathbb{T}$, $\mathbb{P}$-a.s. We write $f(L_t) = \alpha_t$ and $f(U_t) = \beta_t$ for $t \in \mathbb{T}$. Note that $\alpha, \beta \in \mathcal{G}(\mathbb{R})$. Next, we derive the SDE for $(Y_t)_{t\in\mathbb{T}}$, and check the conditions in Definition 2.2 at these boundaries. Using Itô's integration by parts formula, we have the following:

$$\begin{aligned} Y_t &= Y_0 + \int_0^t \frac{\partial f}{\partial x}\, \mathrm{d}X_s^c + \frac{1}{2}\int_0^t \frac{\partial^2 f}{\partial x^2}\, \mathrm{d}\left\langle X_s^c, X_s^c\right\rangle + \sum_{0\le s\le t}\left[f(X_s) - f(X_{s-})\right] \\ &= Y_0 + \int_0^t \left(\frac{\partial f}{\partial x}\mu\left(s, X_s; L_s, U_s\right) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\sigma^2\left(s, X_s; L_s, U_s\right)\right) \mathrm{d}s + \int_0^t \frac{\partial f}{\partial x}\sigma\left(s, X_s; L_s, U_s\right) \mathrm{d}W_s \\ &+ \sum_{0\le s\le t}\left[f(X_s) - f(X_{s-})\right] \Delta J_s \end{aligned} \tag{6}$$

$$\begin{aligned} &\triangleq Y_0 + \int_0^t \hat{\mu}\left(s, Y_s; \alpha_s, \beta_s\right) \mathrm{d}s + \int_0^t \hat{\sigma}\left(s, Y_s; \alpha_s, \beta_s\right) \mathrm{d}W_s \\ &+ \sum_{0\le s\le t} \Delta Y_s, \end{aligned} \tag{7}$$

for all $t \in \mathbb{T}$. We may write $\hat{\mu}$ and $\hat{\sigma}$ in (7) in terms of Y, α and β, since f is strictly monotonic and thus has an inverse (bijective), so that we can find some $\hat{\mu}$ and $\hat{\sigma}$ that can provide Eq. (6). Also, since $f \in C_b^2(\mathbb{R})$ and μ and σ are continuous, $\hat{\mu}$ and $\hat{\sigma}$ are also continuous (and hence locally bounded). It follows that $\Delta Y_t \neq 0$ if and only if $\Delta J_t = 1$ for all $t \in \mathbb{T}$. Since the jump-times of $(J_t)_{t\in\mathbb{T}}$ are mutually independent of $(X_t)_{t\in\mathbb{T}}$, f does not change the distribution of the jump-times of $(Y_t)_{t\in\mathbb{T}}$; it only acts on the jump-sizes. Hence, there exists some function $\hat{\gamma}$ such that

$$\Delta Y_t = \hat{\gamma}\left(t-, Y_{t-}; \alpha_{t-}, \beta_{t-}\right) \Delta J_t,$$

for all $t \in \mathbb{T}$. Therefore, we recover the expression in Eq. (1):

$$Y_t = Y_0 + \int_0^t \hat{\mu}\left(s, Y_s; \alpha_s, \beta_s\right) \mathrm{d}s + \int_0^t \hat{\sigma}\left(s, Y_s; \alpha_s, \beta_s\right) \mathrm{d}W_s + \sum_{0\le s\le t} \hat{\gamma}\left(s-, Y_{s-}; \alpha_{s-}, \beta_{s-}\right) \Delta J_s,$$

for all $t \in \mathbb{T}$. We now need to check whether $\hat{\mu}$ and $\hat{\sigma}$ satisfy Property 1 and Property 2 in Definition 2.2, respectively, and whether $\hat{\gamma}$ satisfies Property 3 in Definition 2.2 at the boundaries. We begin with the case where f is increasing. Then $\alpha < \beta$, and $(Y_t)_{t\in\mathbb{T}}$ attains its minimum at α_t when $X_t = L_t$. Since $\partial f/\partial x > 0$ for any x, we have

$$\hat{\mu}\left(t, \alpha_t; \alpha_t, \beta_t\right) = \frac{\partial f}{\partial L}\mu\left(t, L_t; L_t, U_t\right) \ge \frac{\partial f}{\partial L}\mathrm{d}L_+(t)/\mathrm{d}t = \mathrm{d}\alpha_+(t)/\mathrm{d}t$$

given that $(\partial^2 f/\partial L^2)\sigma^2\left(s, L_s; L_s, U_s\right) = 0$, since $\sigma\left(s, L_s; L_s, U_s\right) = 0$ by Property 2 in Definition 2.2. On the other hand, $(Y_t)_{t\in\mathbb{T}}$ attains its maximum at β_t when $X_t = U_t$, and hence,

$$\hat{\mu}\left(t, \beta_t; \alpha_t, \beta_t\right) = \frac{\partial f}{\partial U}\mu\left(t, U_t; L_t, U_t\right) \le \frac{\partial f}{\partial U}\mathrm{d}U_+(t)/\mathrm{d}t = \mathrm{d}\beta_+(t)/\mathrm{d}t.$$

Therefore, $\hat{\mu}$ satisfies Property 1 in Definition 2.2. As for $\hat{\sigma}$ it satisfies Property 2 in Definition 2.2, since

$$\hat{\sigma}\left(s,\alpha_s;\alpha_s,\beta_s\right)=\frac{\partial f}{\partial L}\sigma\left(s,L_s;L_s,U_s\right)=0 \quad\text{and}\quad \hat{\sigma}\left(s,\beta_s;\alpha_s,\beta_s\right)=\frac{\partial f}{\partial U}\sigma\left(s,U_s;L_s,U_s\right)=0.$$

Finally, $\alpha_{t-}-Y_{t-}\le\hat{\gamma}\left(t-,Y_{t-};\alpha_{t-},\beta_{t-}\right)\le\beta_{t-}-Y_{t-}$ must hold $\mathbb{P}$-a.s., since $\alpha_t\le Y_t\le\beta_t$ holds and $\Delta J_t\in\{0,1\}$ for all $t\in\mathbb{T}$, $\mathbb{P}$-a.s. The case when f is decreasing follows similarly. Here, $\beta<\alpha$, and $(Y_t)_{t\in\mathbb{T}}$ attains its minimum at β_t when $X_t=U_t$. Hence, having $\partial f/\partial x<0$ for any x,

$$\hat{\mu}\left(t,\beta_t;\alpha_t,\beta_t\right)=\frac{\partial f}{\partial U}\mu\left(t,U_t;L_t,U_t\right)\ge\frac{\partial f}{\partial U}\mathrm{d}U_+(t)/\mathrm{d}t=\mathrm{d}\beta_+(t)/\mathrm{d}t$$

Also, since$(Y_t)_{t\in\mathbb{T}}$ attains its maximum at α_t when $X_t=L_t$, we have

$$\hat{\mu}\left(t,\alpha_t;\alpha_t,\beta_t\right)=\frac{\partial f}{\partial L}\mu\left(t,L_t;L_t,U_t\right)\le\frac{\partial f}{\partial L}\mathrm{d}L_+(t)/\mathrm{d}t=\mathrm{d}\alpha_+(t)/\mathrm{d}t.$$

As before, $\hat{\mu}$ satisfies Property 1 in Definition 2.2. We already know $\hat{\sigma}$ satisfies Property 2 in Definition 2.2. Finally, since $\beta_t\le Y_t\le\alpha_t$ for all $t\in\mathbb{T}$, $\mathbb{P}$-a.s., we must have $\beta_{t-}-Y_{t-}\le\hat{\gamma}\left(t-,Y_{t-};\alpha_{t-},\beta_{t-}\right)\le\alpha_{t-}-Y_{t-}$ for all $t\in\mathbb{T}$, $\mathbb{P}$-a.s., which is Property 3 in Definition 2.2. □

Example 2.12. Let $Y_t=e^{X_t}$ for all $t\in\mathbb{T}$. Then, $(Y_t)_{t\in\mathbb{T}}$ is a captive jump process, where

$$e^{g^l(t)}\le Y_t\le e^{g^u(t)}\qquad\text{for all } t\in\mathbb{T},\ \mathbb{P}\text{-a.s.}$$

Example 2.13. Let $Y_t=(X_t+c)^n$ be a polynomial for some $0\le c<\infty$ and $1\le n<\infty$, for all $t\in\mathbb{T}$. Set $0<g^l(t)$ for all $t\in\mathbb{T}$. Then, $(Y_t)_{t\in\mathbb{T}}$ is a captive jump process, where

$$(g^l(t)+c)^n\le Y_t\le(g^u(t)+c)^n\qquad\text{for all } t\in\mathbb{T},\ \mathbb{P}\text{-a.s.}$$

Example 2.14. Let $0<g^l(t)$ for all $t\in\mathbb{T}$ and $Y_t=X_t^{-1}$ for all $t\in\mathbb{T}$. Then, $(Y_t)_{t\in\mathbb{T}}$ is a captive jump process, where

$$g^u(t)^{-1}\le Y_t\le g^l(t)^{-1}\qquad\text{for all } t\in\mathbb{T},\ \mathbb{P}\text{-a.s.}$$

Since continuous bijective functions are strictly monotonic, we have the following corollary.

Corollary 2.15. *Let $\mathcal{X}$ and $\mathcal{Y}$ be compact sets in $\mathbb{R}$, where $\mathcal{X}$ is the domain of $(X_t)_{t\in\mathbb{T}}$. Let the map $f:\mathcal{X}\to\mathcal{Y}$ be bijective, where $f\in C_b^2(\mathbb{R})$ and $Y_t=f(X_t)$ for all $t\in\mathbb{T}$. Then $(Y_t)_{t\in\mathbb{T}}$ is a captive jump process.*

Example 2.16. Let $\mathcal{X}=[-\pi/2,\pi/2]$ and $\mathcal{Y}=[-1,1]$, where $f:\mathcal{X}\to\mathcal{Y}$ is $f(x)=\sin(x)$. Let $(X_t)_{t\in\mathbb{T}}$ be a continuous captive martingale governed by

$$X_t=x_0+\int_0^t\sin\left(X_s-\frac{\pi}{2}\right)\sin\left(X_s+\frac{\pi}{2}\right)\mathrm{d}W_s,$$

where $x_0\in(-\pi/2,\pi/2)$. Using Itô's integration by parts formula, we have

$$\begin{aligned}Y_t&=y_0+\int_0^t\cos(X_s)\sin\left(X_s-\frac{\pi}{2}\right)\sin\left(X_s+\frac{\pi}{2}\right)\mathrm{d}W_s-\frac{1}{2}\sin(X_s)\sin\left(X_s-\frac{\pi}{2}\right)^2\sin\left(X_s+\frac{\pi}{2}\right)^2\mathrm{d}s\\&=y_0-\int_0^t\cos^3(X_s)\,\mathrm{d}W_s-\frac{1}{2}\sin(X_s)\cos^4(X_s)\,\mathrm{d}s\\&=y_0-\int_0^t\cos^3(\sin^{-1}(Y_s))\,\mathrm{d}W_s-\frac{1}{2}Y_s\cos^4(\sin^{-1}(Y_s))\,\mathrm{d}s\\&=y_0-\int_0^t\left(1-Y_s^2\right)^{\frac{3}{2}}\mathrm{d}W_s-\frac{1}{2}Y_s\left(1-Y_s^2\right)^2\mathrm{d}s\end{aligned}$$

which shows that $(Y_t)_{t\in\mathbb{T}}$ is a continuous captive diffusion process with absorbing boundaries at levels -1 and 1.

Before we introduce path-dependency and internal corridors into the framework, we shall briefly look at captive jump processes from the perspective of partial differential equations.

Proposition 2.17. *Let $(\lambda_t)_{t\in\mathbb{T}}$ be the intensity process of $(J_t)_{t\in\mathbb{T}}$, $\psi:\mathbb{R}\to\mathbb{R}$ and $\phi:\mathbb{R}\to\mathbb{R}$ be continuous bounded functions, and*

$$v(x,t):=\mathbb{E}\left[e^{-\int_t^T\psi(s)\mathrm{d}s}\phi(X_T)\,\middle|\,X_t=x\right].$$

If $v\in C^2(\mathbb{R})$, then

$$\begin{aligned}&\frac{\partial v(g^l(t),t)}{\partial t}+\mu(t,g^l(t);g^l(t),g^u(t))\frac{\partial v(g^l(t),t)}{\partial g^l(t)}+(v(g^l(t),t)-v(g^l(t-),t-))\lambda_t=\psi(t)v(g^l(t),t)\\&\frac{\partial v(g^u(t),t)}{\partial t}+\mu(t,g^u(t);g^l(t),g^u(t))\frac{\partial v(g^u(t),t)}{\partial g^u(t)}+(v(g^u(t),t)-v(g^u(t-),t-))\lambda_t=\psi(t)v(g^u(t),t),\end{aligned}$$

where $v(x,T)=\phi(x)$ for all $x\in[g^l(T),g^u(T)]$.

Proof. Using the Doob–Meyer decomposition, we have $J_t = \hat{J}_t + \Lambda_t$ for all $t \in \mathbb{T}$, where $(\hat{J}_t)_{t\in\mathbb{T}}$ is a martingale and $\Lambda_t = \int_0^t \lambda_s \mathrm{d}s$ is the compensator. In addition,

$$\hat{v}(X_t, t) = e^{-\int_0^t \psi(s)\mathrm{d}s}\, v(X_t, t)$$

defines a martingale process $(\hat{v}(X_t, t))_{t\in\mathbb{T}}$. Applying Itô's integration by parts formula to $\hat{v}(X_t, t)$, we have

$$\begin{aligned}&\frac{\partial \hat{v}(x,t)}{\partial t}\mathrm{d}t + \frac{\partial \hat{v}(x,t)}{\partial x}\left(\sigma(t,x;g^l(t),g^u(t))\mathrm{d}W_t + \mu(t,x;g^l(t),g^u(t))\mathrm{d}t\right)\\ &\quad + \frac{1}{2}\frac{\partial^2 \hat{v}(x,t)}{\partial x^2}\sigma^2(t,x;g^l(t),g^u(t))\mathrm{d}t + (\hat{v}(x,t) - \hat{v}(x-,t-))\,(\mathrm{d}\hat{J}_t + \mathrm{d}\Lambda_t).\end{aligned}$$

Since $(\hat{v}(X_t, t))_{t\in\mathbb{T}}$ is a martingale, we must thus have

$$\begin{aligned}&e^{-\int_0^t \psi(s)\mathrm{d}s}\left(\frac{\partial v(x,t)}{\partial t} - \psi(t)v(x,t) + \frac{\partial v(x,t)}{\partial x}\mu(t,x;g^l(t),g^u(t)) + \frac{1}{2}\frac{\partial^2 \hat{v}(x,t)}{\partial x^2}\sigma^2(t,x;g^l(t),g^u(t))\right)\mathrm{d}t\\ &+ e^{-\int_0^t \psi(s)\mathrm{d}s}(v(x,t) - e^{-\int_0^{t-} \psi(s)\mathrm{d}s}v(x-,t-))\lambda_t\mathrm{d}t = 0,\end{aligned}$$

Dividing both sides by $e^{-\int_0^t \psi(s)\mathrm{d}s}\mathrm{d}t$, one obtains

$$\left(v(x,t) - e^{\int_{t-}^t \psi(s)\mathrm{d}s}v(x-,t-)\right)\lambda_t = (v(x,t) - v(x-,t-))\,\lambda_t,$$

since $\int_{t-}^t \psi(s)\mathrm{d}s = 0$ due to the continuity property of ψ. Therefore, we have

$$\begin{aligned}&\frac{\partial v(x,t)}{\partial t} - \psi(t)v(x,t) + \frac{\partial v(x,t)}{\partial x}\mu(t,x;g^l(t),g^u(t)) + \frac{1}{2}\frac{\partial^2 \hat{v}(x,t)}{\partial x^2}\sigma^2(t,x;g^l(t),g^u(t))\\ &+(v(x,t) - v(x-,t-))\lambda_t = 0\end{aligned}$$

Using Property 2 in Definition 2.2, we have $\sigma^2(t,g^l(t);g^l(t),g^u(t)) = \sigma^2(t,g^u(t);g^l(t),g^u(t)) = 0$, and the statement is proven. □

Corollary 2.18. *Given the setup in Proposition 2.17, where $\psi(t) = 0$ for all $t \in \mathbb{T}$, $(X_t)_{t\in\mathbb{T}}$ is continuous. Then,*

$$\mu(t,g^l(t);g^l(t),g^u(t))\frac{\partial v(g^l(t),t)}{\partial g^l(t)} = \mu(t,g^u(t);g^l(t),g^u(t))\frac{\partial v(g^u(t),t)}{\partial g^u(t)} = 0.$$

2.2. Path-dependent captive jump processes

So far, the generated jump processes have been restricted to never leave a single confined space. In [4] the so-called *tunnelled* captive diffusions are introduced, which are confined to stay within a finite space (or an internal corridor) for some time before being allowed to diffuse into another internal confined subspace in a following period of time; while always remaining within their master boundaries. This behaviour makes of a captive diffusion a path-dependent process, because of the implicit monitoring that is required in order to control the shift of the diffusion from one internal corridor to another. In what follows, we show how CJPs can be constructed such that for part of the time they stay confined in a bounded subspace before they *jump* to another restricted area for some time. This behaviour requires the modelling of controlled jumps (arrival times and sizes) and historic monitoring of the process, which in turn makes the CJP path-dependent. As Section 3 will illustrate, the restricted areas (corridors) captive jump processes may jump to are not required to be adjacent, in the sense that the process may skip neighbouring corridors. Such features render CJPs flexible and useful in a variety of applications, see Section 3.

Next, we extend Definition 2.2 to allow for *internal* boundaries to appear. In doing so, we proceed in the spirit of [4], and introduce multiple time-segments denoted by $\mathbb{T}^{(j)} \subseteq \mathbb{T}$ for $j = 0,\dots,m \in \mathbb{N}_+$, for a fixed $1 \le m < \infty$. Here,

$$\mathbb{T}^{(j)} = [\tau^{(j)}_{\text{start}}, \tau^{(j)}_{\text{end}}] \qquad \text{such that} \qquad 0 \le \tau^{(j)}_{\text{start}} < \tau^{(j)}_{\text{end}} \le T,$$

for $j = 0,\dots,m$. We choose $\mathbb{T}^{(0)} = \mathbb{T}^{(m)} = \mathbb{T}$, that is, $\tau^{(0)}_{\text{start}} = \tau^{(m)}_{\text{start}} = 0$ and $\tau^{(0)}_{\text{end}} = \tau^{(m)}_{\text{end}} = T$. Each of these time-segments govern a boundary process $(g^{(j)}_t)_{t\in\mathbb{T}^{(j)}}$, where we have $g^{(j)} \in \mathcal{G}(\mathbb{R})$ for $j = 0,\dots,m$. We emphasize that $(g^{(0)}_t)_{t\in\mathbb{T}^{(0)}}$ and $(g^{(m)}_t)_{t\in\mathbb{T}^{(m)}}$ are the master, or outermost, boundaries. In addition, we collect all the boundaries in the set-valued process $(\mathbf{g}_t)_{t\in\mathbb{T}}$ given by

$$\mathbf{g}_t = \{g^{(0)}_t, \dots, g^{(m)}_t\}.$$

In the situation where there is a $g^{(j)}_t$, for $0 < j < m$, that is not defined over some $t \in \mathbb{T}$, then $\mathbf{g}_t$ does not include such a $g^{(j)}_t$ at time $t \in \mathbb{T}$. Finally, for any $j < k$ at any $t \in \mathbb{T}^{(j)} \cap \mathbb{T}^{(k)} \neq \emptyset$, we require the ordering $g^{(j)}_t < g^{(k)}_t$, and for any pair $\{g^{(j)}_t, g^{(k)}_t\}$ where $\mathbb{T}^{(j)} \cap \mathbb{T}^{(k)} = \emptyset$, their order does not matter amongst each other.

Remark 2.19. The *master boundaries* $g^{(0)}$ and $g^{(m)}$ define the largest bounded domain on which captive jump processes may evolve, and each $g^{(j)}$ is called an *internal boundary*, for $j \neq 0$, $j \neq m$ given that $m > 1$. Clearly, $\mathbb{T}^{(j)} \subseteq \mathbb{T}$ for every $j = 0,\dots,m$.

Next, we introduce progressively-measurable and increasing processes $(\tau^{(j)}_t)_{t\in\mathbb{T}^{(j)}}$ given by

$$\tau^{(j)}_t = \tau^{(j)}_{\text{start}} \vee \sup(s : \Delta X_s \neq 0 \quad \text{for} \quad \tau^{(j)}_{\text{start}} \le s \le t \in \mathbb{T}^{(j)}), \tag{8}$$

for $j = 0, \dots, m-1$, where we adopt the convention $\sup \emptyset = -\infty$. Hence, if there is no jump in a given time period $\mathbb{T}^{(j)}$, then $\tau_t^{(j)} = \tau_{\text{start}}^{(j)}$ for all t in the period $\mathbb{T}^{(j)}$. Note that since $\mathbb{T}^{(0)} = \mathbb{T}^{(m)} = \mathbb{T}$, we have $\tau_t^{(0)} = \tau_t^{(m)}$ for every $t \in \mathbb{T}$, which is the reason why we can exclude m from the definition of $\tau_t^{(j)}$ in (8).

Remark 2.20. We shall highlight the importance of the progressively-measurable process $(\tau_t^{(j)})_{t\in\mathbb{T}^{(j)}}$ in Eq. (8) for Definition 2.21, Proposition 2.22 below. Monitoring $(\tau_t^{(j)})_{t\in\mathbb{T}^{(j)}}$ is necessary to ensure that $(X_t)_{t\in\mathbb{T}}$ stays in the most recently visited internal corridor it may have jumped to. This aspect is fleshed-out in what follows.

We now introduce a monitoring process $(\Psi_t)_{t\in\mathbb{T}}$ that records the values of $(X_t)_{t\in\mathbb{T}}$ at $(\tau_t^{(j)})_{t\in\mathbb{T}^{(j)}}$ for $j = 0, \dots, m$. As such, we let $(\Psi_t)_{t\in\mathbb{T}}$ be the non-anticipative and set-valued process given by

$$\Psi_t = \{X_{\tau_t^{(j)}},\, \tau_t^{(j)} \,:\, \tau_t^{(j)} \le t, \text{ for } j = 0, \dots, m-1\},$$

for all $t \in \mathbb{T}$. In the case that we require a function to be continuous with respect to Ψ, we mean that the function is continuous with respect to the elements of Ψ. Since Ψ_t is a set-valued random variable for each $t \in \mathbb{T}$, we clarify what is meant by its measurability property. We note that Ψ_t is a compact set for every $t \in \mathbb{T}$ since $m < \infty$. Hence, denoting by $\mathcal{X}(\mathbb{R})$ the family of compact subsets of $\mathbb{R}$, the map $\Psi : \Omega \times \mathbb{T} \to \mathcal{X}(\mathbb{R})$ defines a compact set-valued random variable, where its measurability condition is given by

$$\{(\omega, t) \in \Omega \times \mathbb{T} : \Psi_t(\omega) \cap A \neq \emptyset\} \in \mathcal{F} \qquad \forall A \in \mathcal{B}(\mathbb{R}), \quad \Psi_t(\omega) \in \mathcal{X}(\mathbb{R}). \tag{9}$$

For the notion of measurability of set-valued random variables, we refer the reader to [32–36]. For the process $(\Psi_t)_{t\in\mathbb{T}}$ to be adapted, $(\Psi_t)_{t\in\mathbb{T}}$ needs to be progressively measurable, that is

$$(\omega, t) \mapsto \Psi_t(\omega) \quad \text{is} \quad \left(\mathcal{F}_t^X \otimes \mathcal{B}(\mathbb{T})\right)\text{-measurable} \quad \forall t \in \mathbb{T}, \tag{10}$$

in the above sense. The following definition allows us to introduce internal corridors and may also be viewed as a lemma to Proposition 2.22, see below.

Definition 2.21. An internally piecewise-confined captive jump process $(X_t)_{t\in\mathbb{T}}$ is a solution to the SDE

$$X_t = x_0 + \int_0^t \mu\left(s, \Psi_s, X_s; \mathbf{g}_s\right) \mathrm{d}s + \int_0^t \sigma\left(s, \Psi_s, X_s; \mathbf{g}_s\right) \mathrm{d}M_s + \sum_{0\le s\le t} \gamma\left(s-, \Psi_{s-}, X_{s-}; \mathbf{g}_{s-}\right) \Delta J_s, \tag{11}$$

where $X_0 = x_0 \in [g_0^{(0)}, g_0^{(m)})$. The maps μ and σ are continuous (possibly except at points where $\Delta J \neq 0$ with bounded jumps), and γ is a locally bounded càdlàg map such that

1. $\mu\left(t, \Psi_t, g_t^{(j)}; \mathbf{g}_t\right) \ge \mathrm{d}g_+^{(j)}(t)/\mathrm{d}t$, if $X_{\tau_t^{(j)}} \ge g_{\tau_t^{(j)}}^{(j)}$ for any $t \in \mathbb{T}^{(j)}$ where $X_t = g_t^{(j)}$;
2. $\mu\left(t, \Psi_t, g_t^{(j)}; \mathbf{g}_t\right) \le \mathrm{d}g_+^{(j)}(t)/\mathrm{d}t$, if $X_{\tau_t^{(j)}} < g_{\tau_t^{(j)}}^{(j)}$ for any $t \in \mathbb{T}^{(j)}$ where $X_t = g_t^{(j)}$;
3. $\sigma\left(t, \Psi_t, g_t^{(j)}; \mathbf{g}_t\right) = 0$ for any $t \in \mathbb{T}^{(j)}$ where $X_t = g_t^{(j)}$;
4. $g_{t-}^{(0)} - X_{t-} \le \gamma\left(t-, \Psi_{t-}, X_{t-}; \mathbf{g}_{t-}\right) \le g_{t-}^{(m)} - X_{t-}$ for all $t \in \mathbb{T}$,

for $j = 0, \dots, m$ $\mathbb{P}$-a.s., given that $(M_t)_{t\in\mathbb{T}} \in \mathcal{M}(\mathbb{R})$ and $(J_t)_{t\in\mathbb{T}} \in \mathcal{J}(\mathbb{R})$ are mutually independent.

There are a few observations worth making at this stage which will help with the proof below. First, we notice that $\Delta g_t^{(j)} = 0$ for any $j \neq 0$ and $j \neq m$, since they belong to $\mathcal{G}$, i.e., internal boundaries cannot have discontinuities. This however is no real restriction, since multiple internal confining functions $(g_t^{(j)})_{t\in\mathbb{T}}$ can be used in sequence to construct a piecewise process that behaves like an internal corridor with jumps. The reason for requiring each $g^{(j)} \in \mathcal{G}$ for any $j \neq 0$ and $j \neq m$ is a technical requirement that is needed for Proposition 2.22 to ensure captivity in *each* corridor segment that would otherwise be broken over time periods where $(X_t)_{t\in\mathbb{T}}$ is continuous. Second, the SDE coefficients are path-dependent, that is, they monitor past values of the CJP at each $(\tau_t^{(j)})_{t\in\mathbb{T}^{(j)}}$. This means that such processes can be non-Markovian even if $(M_t)_{t\in\mathbb{T}}$ and $(J_t)_{t\in\mathbb{T}}$ are Markov processes. Third, the initial condition $x_0 \in [g_0^{(0)}, g_0^{(m)})$ does not include $g_0^{(m)}$, since we associated the case $X(\tau_{\text{start}}^{(j)}) = g^{(j)}(\tau_{\text{start}}^{(j)})$ with Property 1 above that, alternatively, could have been associated with Property 2 having initial condition $x_0 \in (g_0^{(0)}, g_0^{(m)}]$. Fourth, Property 4 only requires the jump coefficient γ to be bounded with respect to the *master* boundaries $\{g_t^{(0)}, g_t^{(m)}\}$. Hence, Definition 2.21 allows for jump-sizes to take $(X_t)_{t\in\mathbb{T}}$ from one *internal* corridor to another. We are now in the position to state the following result:

Proposition 2.22. *The following statements hold $\mathbb{P}$-almost surely.*

1. *For any $t \in \mathbb{T}$, $g_t^{(0)} \le X_t \le g_t^{(m)}$.*
2. *For any $j \neq 0$ and $j \neq m$, if $X_{\tau_{start}^{(j)}} \ge g_{\tau_{start}^{(j)}}^{(j)}$ and $\Delta J_t = 0$ for all $t \in \mathbb{T}^{(j)}$, then $X_t \ge g_t^{(j)}$ for all $t \in \mathbb{T}^{(j)}$. If $\Delta J_t = 1$ for some $t \in \mathbb{T}^{(j)}$ then $\mathbb{P}(X_t < g_t^{(j)}) \ge 0$ for $t \in \mathbb{T}^{(j)}$.*
3. *For any $j \neq 0$ and $j \neq m$, if $X_{\tau_{start}^{(j)}} < g_{\tau_{start}^{(j)}}^{(j)}$ and if $\Delta J_t = 0$ for all $t \in \mathbb{T}^{(j)}/(\tau_{start}^{(j)})$, then $X_t \le g_t^{(j)}$ for all $t \in \mathbb{T}^{(j)}/(\tau_{start}^{(j)})$. If $\Delta J_t = 1$ for some $t \in \mathbb{T}^{(j)}$ then $\mathbb{P}(X_t > g_t^{(j)}) \ge 0$ for $t \in \mathbb{T}^{(j)}/(\tau_{start}^{(j)})$*

Proof. This follows from the construction given by Definition 2.21 together with Proposition 2.3, and Proposition 2.8 in [4], combined with Property 4 in Definition 2.21: it allows for ΔX_t, $t \in \mathbb{T}$, to be large enough for $(X_t)_{t\in\mathbb{T}}$ to exceed the boundary $g^{(j)}$ for $j \neq 0$ and $j \neq m$, but not the master boundaries $\{g^{(0)}, g^{(m)}\}$, $\mathbb{P}$-a.s. The progressively measurable process $(\tau_t^{(j)})_{t\in\mathbb{T}^{(j)}}$ in (8) ensures that the ordering of the right-derivatives agree with the direction of $(X_t)_{t\in\mathbb{T}}$ at the boundaries so that $g_t^{(0)} \leq X_t \leq g_t^{(m)}$ holds for every $t \in \mathbb{T}$, as well as the captivity within any internal corridor when there is no jump over the corresponding time horizon $\mathbb{T}^{(j)}$. □

Proposition 2.22 tells us that if there is no jump during an internal corridor, $(X_t)_{t\in\mathbb{T}}$ cannot leave that corridor. It also tells us that γ can be chosen in such a way that if $(X_t)_{t\in\mathbb{T}}$ is in an internal corridor, then it may still stay in that corridor even if there is a (small enough) jump. Property 4 of Definition 2.21 allows for this feature. However, we are more interested in modelling systems where a transition between two internal corridors is allowed if there is a jump, and *only* if there is a jump, as Definition 2.21 offers. Many of the results from the previous section can be extended to piecewise-confined CJPs, which we omit to avoid repetition.

3. Applications

For all the examples below, the coefficients satisfy local Lipschitz-continuity (possibly except at points where $\Delta J \neq 0$, see for example [37] for SDEs with piecewise-continuous coefficients), and each SDE below has a bounded solution. We expect many other examples can be constructed and studied in detail by using Definition 2.21. For the simulations, we use the Euler–Maruyama scheme. If $\{\hat{X}_{t_k}\}_{t_k\in\mathbb{T}}$ is an approximation of $(X_t)_{t\in\mathbb{T}}$ over the discretized grid $0 = t_0 \leq t_1 \leq \cdots \leq t_m \leq= T < \infty$ for some $m \in \mathbb{N}_+$, then the path-by-path simulation scheme is generated as follows:

1. For $t = t_0$, set $\hat{X}_t = x_0$ and $\Delta J_t = 0$.
2. For $t = t_k$, let
   $\hat{X}_{t_{k+1}} = \hat{X}_t \,+\, \mu\left(t,\Psi_t,\hat{X}_t;g_t\right)\delta \,+\, \sigma\left(t,\Psi_t,\hat{X}_t;g_t\right)\left(W_{t_{k+1}} - W_t\right) + A_t$, where
   $A_t = \gamma\left(t,\Psi_t,\hat{X}_t;g_t\right)$ if $\Delta J_t = 1$, otherwise $A_t = 0$.
3. Repeat Step 2 for $k = 1, \ldots, m-1$,

where $\delta = T/m$, $t_k = k\delta$, and $\Delta J_{t_k} \in \{0,1\}$ for every $t_k \in \mathbb{T}$. Usually, the weak and strong convergence of numerical methods for SDEs rely on global Lipschitz-continuity, see [38], where the pathwise error for $(\hat{X}_{t_k})_{t_k\in\mathbb{T}}$ is given by

$$\sup_{k=0,\ldots,m} |\hat{X}_{t_k}(\omega) - X_{t_k}(\omega)|.$$

For Itô-Taylor schemes, such as the Euler–Maruyama scheme, this condition can be relaxed to local Lipschitz-continuity, where

$$\sup_{k=0,\ldots,m} \left|\hat{X}_{t_k}(\omega) - X_{t_k}(\omega)\right| \leq \phi_\epsilon(\omega) m^{\epsilon-1/2},$$

for all $\epsilon > 0$ and almost all $\omega \in \Omega$, where $\phi_\epsilon : \Omega \to \mathbb{R}_+$ is finite, such that the pathwise order of convergence equals $(1/2-\epsilon)$, see [39,40]. Since CJPs in general do not admit closed-form representations, the use of Euler–Maruyama scheme does not guarantee all simulated paths to remain within (potentially complex) bounded domains. We shall run a numerical sensitivity analysis with respect to different choices of δ for different applications that we demonstrate below. We later present a modified Euler–Maruyama scheme that adjusts for path violations in an implicit way.

We also explain how path-dependency arising through the monitoring process $(\Psi_t)_{t\in\mathbb{T}}$ can be implemented. Since $m < \infty$ and $(J_t)_{t\in\mathbb{T}}$ is of finite-activity, Ψ_t is a finite set for every $t \in \mathbb{T}$. Thus, in all produced examples, both μ and σ satisfy

$$\mu\left(t,\Psi_t,X_t;\mathbf{g}_t\right) = \mu\left(t, X_{\tau_t^{(0)}}, \tau_t^{(0)}, \ldots, X_{\tau_t^{(j^*)}}, \tau_t^{(j^*)}, X_t;\mathbf{g}_t\right),$$
$$\sigma\left(t,\Psi_t,X_t;\mathbf{g}_t\right) = \sigma\left(t, X_{\tau_t^{(0)}}, \tau_t^{(0)}, \ldots, X_{\tau_t^{(j^*)}}, \tau_t^{(j^*)}, X_t;\mathbf{g}_t\right)$$

with respect to Ψ_t for every $t \in \mathbb{T}$ where

$$j^* = \max\{\, j \;:\; \tau_{\text{start}}^{(j)} \leq t \,,\, 0 < j < m\}.$$

The property possessed by the functions μ and σ, c.f. above, is also enjoyed by the jump size function γ. The numerical simulations generated next handle path-dependency through μ, σ, and γ which are projected onto higher-dimensional spaces. This will be demonstrated explicitly through the provided path-dependent examples in which we will use conditional *if arguments* as a function of time. This can be represented mathematically by using a collection of indicator functions.

3.1. Confinement within circular domains

We first consider a setup without internal corridors. Let $(X_t^{(1)})_{t\in\mathbb{T}}$ be a captive jump–diffusion, where $X_0^{(0)} \in [a,d]$ for some $0 \leq a < d < \infty$. Let $(X_t^{(2)})_{t\in\mathbb{T}}$ be a second captive jump–diffusion where $X_0^{(0)} \in [0, 2\pi]$. These two processes are governed by

$$\mathrm{d}X_t^{(i)} = (\beta^{(i)} - X_t^{(i)})\mathrm{d}t + (X_t^{(i)} - L^{(i)})\left(U^{(i)} - X_t^{(i)}\right)\mathrm{d}W_t^{(i)} + \theta_{t-}^{(i)} \min\left(X_{t-}^{(i)} - L^{(i)}, U^{(i)} - X_{t-}^{(i)}\right)\Delta J_t^{(i)}.$$

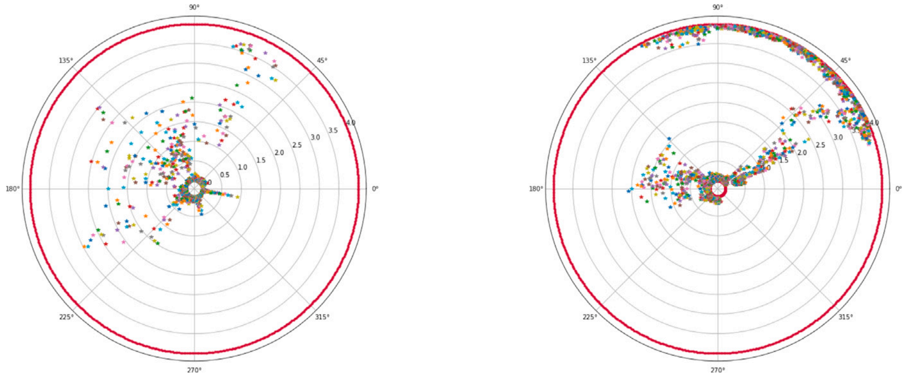

Fig. 2. Here, $a = 0$, $d = r = 4$ and $\beta = \frac{1}{2}(a + d)$.

Here, the Brownian motions $(W_t^{(1)})_{t\in\mathbb{T}}$ and $(W_t^{(2)})_{t\in\mathbb{T}}$ may be correlated. Similarly, $(J_t^{(1)})_{t\in\mathbb{T}}$ and $(J_t^{(2)})_{t\in\mathbb{T}}$ may also be correlated. The process $(\theta_t)_{t\in\mathbb{T}}$ may be any càdlàg map as long as $\theta_t \in [-1, 1]/\{0\}$ for all $t \in \mathbb{T}$. For the examples below, we simulate in advance a random path for $(\theta_t)_{t\in\mathbb{T}}$ by uniformly sampling its values at every step on a discrete time grid. The lower and upper boundaries are given by

$$a = L^{(1)} < \beta^{(1)} < U^{(1)} = d$$
$$0 = L^{(2)} < \beta^{(2)} < U^{(2)} = 2\pi$$

respectively. Next, we construct a two-dimensional process $(P_t)_{t\in\mathbb{T}}$ given by $(X_t^{(1)}, X_t^{(2)})_{t\in\mathbb{T}}$ on a polar coordinate system, where $(X_t^{(1)})_{t\in\mathbb{T}}$ models the distance from the origin and $(X_t^{(2)})_{t\in\mathbb{T}}$ is the radian process. First we show the case where an outer circle serves as the outer boundary for the confined circular domain. One can see two different *accumulation* behaviours in Fig. 2. On the left-hand side, we see that the process tends to evolve towards the origin of the circle, while on the right-hand side, the process visits the boundary of the circular domain many times during the simulation period.

Remark 3.1. For the case of central accumulation, the drift and volatility of the distance process $(X_t^{(1)})_{t\in\mathbb{T}}$ can be used to model the gravitational force exerted on jump-diffusing particles by a central mass sitting at the origin.

Remark 3.2. One can introduce a third captive jump–diffusion process $(X_t^{(3)})_{t\in\mathbb{T}}$ as the second orthogonal radian coordinate, and project $(X_t^{(1)}, X_t^{(2)}, X_t^{(3)})_{t\in\mathbb{T}}$ inside a sphere. This indicates how the construction of CJPs can be extended to processes taking values in confined domains in $\mathbb{R}^n$, $n \in \mathbb{N}$.

In Fig. 3, we shrink the domain inside the circle to keep the paths within shrinking rings, each shaped as a toroid. Again, this can be extended to three dimensions in the spirit of Remark 3.2.

This setup can be used to model systems with a central gravitational force which keeps stochastic particles within circular corridors. Here, even if $(X_t^{(1)}, X_t^{(2)})_{t\in\mathbb{T}}$ jumps, it can never cross the master (outer) boundaries and break free from the confined toroidal space. In the next section, we shall add internal corridors, which the particles are allowed to trespass only if they jump far enough.

3.2. Captive jumps across circular domains

Now we consider the situation where, in addition to the outer boundaries, there are two inner boundaries for the distance-process $(X_t^{(1)})_{t\in\mathbb{T}}$. We keep the radian-process $(X_t^{(2)})_{t\in\mathbb{T}}$ as in the previous section. We set $\mathbb{T}^{(0)} = \mathbb{T}^{(1)} = \mathbb{T}^{(2)} = \mathbb{T}^{(3)} = \mathbb{T}$ such that

$$(g_t^{(0)})_{t\in\mathbb{T}} = a, \quad (g_t^{(1)})_{t\in\mathbb{T}} = b, \quad (g_t^{(2)})_{t\in\mathbb{T}} = c, \quad (g_t^{(3)})_{t\in\mathbb{T}} = d.$$

Hence, $\tau_{\text{start}}^{(j)} = 0$ and $\tau_{\text{end}}^{(j)} = T$ for $j = 1, \dots, 3$. So, we write $\tau_t^{(j)} = \tau_t$ for $j = 1, \dots, 3$ and $t \in \mathbb{T}$. We initialize $(X_t^{(1)})_{t\in\mathbb{T}}$ such that it starts within the inner circular corridor, where $X_0^{(0)} \in [a, b]$ for some $0 \le a < b < c < d < \infty$, where $[a, b]$ forms the innermost corridor, $[b, c]$ forms a mid-corridor and $[c, d]$ forms the outermost corridor for $(X_t^{(1)})_{t\in\mathbb{T}}$, which is now governed by

$$\mathrm{d}X_t^{(1)} = (\beta_t^{(1)}(\Psi_t^{(1)}) - X_t^{(1)})\mathrm{d}t + \prod_{g^{(j)}\in \boldsymbol{g}} (X_t^{(1)} - g^{(j)})\mathrm{d}W_t^{(1)} + \theta_{t-}^{(1)} \min\left(X_{t-}^{(1)} - a, d - X_{t-}^{(1)}\right) \Delta J_t^{(1)}.$$

Here, $\boldsymbol{g} = \{a, b, c, d\}$, and $(\beta_t^{(1)}(\Psi_t^{(1)}))_{t\in\mathbb{T}}$ is given by

$$\beta_t^{(1)}(\Psi_t^{(1)}) = \begin{cases} w_1 a + (1 - w_1)b & \text{if } X_{\tau_t} \in [a, b), \\ w_2 b + (1 - w_2)c & \text{if } X_{\tau_t} \in [b, c), \\ w_3 c + (1 - w_3)d & \text{if } X_{\tau_t} \in [c, d], \end{cases}$$

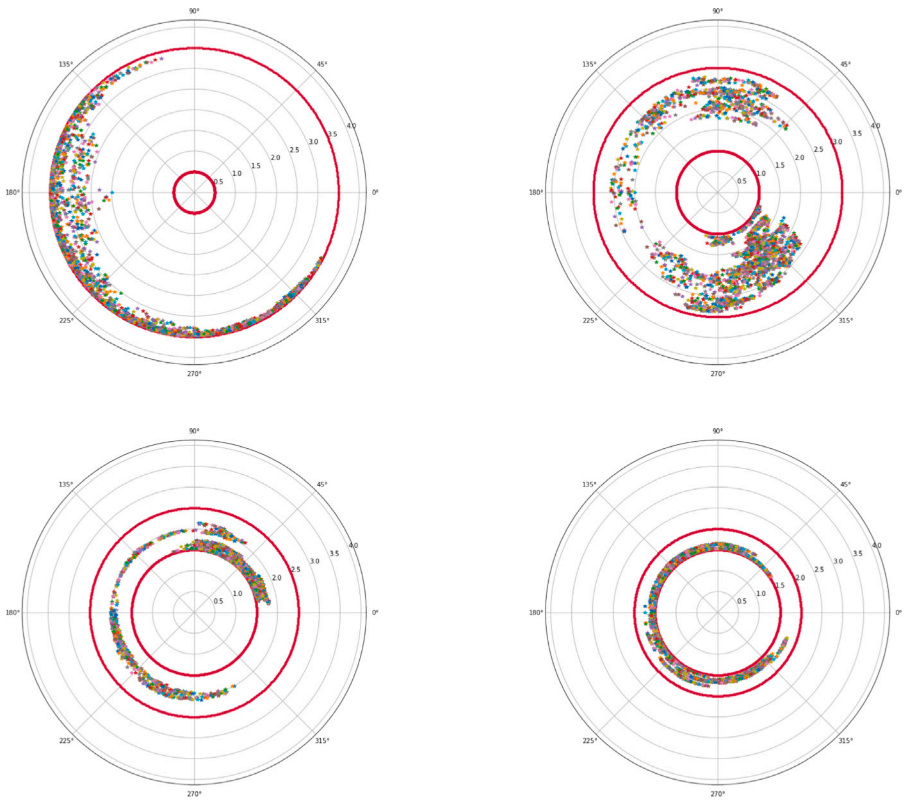

Fig. 3. The boundaries are: Top-left {0, 3.5}, top-right {1, 3}, bottom-left {1.5, 2.5}, and bottom-right {1.5, 2}.

for some $w_1, w_2, w_3 \in (0,1)$. Next, we construct a process $(P_t)_{t\in\mathbb{T}}$ by introducing the two-dimensional process $(X_t^{(1)}, X_t^{(2)})_{t\in\mathbb{T}}$ taking values in the polar coordinate system. The plots in Fig. 4 show samples of $(X_t^{(1)})_{t\in\mathbb{T}}$ and the associated process $(X_t^{(1)}, X_t^{(2)})_{t\in\mathbb{T}}$. The simulation on the left-hand side shows how $(X_t^{(1)}, X_t^{(2)})_{t\in\mathbb{T}}$ in polar coordinates stays within the inner or outer corridors, where the transition from $[a,b)$ to $[c,d]$ occurs when $(X_t^{(1)})_{t\in\mathbb{T}}$ jumps far enough to skip the mid-corridor $[b,c)$. On the right-hand side, $(X_t^{(1)}, X_t^{(2)})_{t\in\mathbb{T}}$ visits every corridor, depending on the size of the jumps. Of course, the process might jump within a specific corridor without necessarily leaving it.

The conditional state probabilities can be calculated, where in our context, *state* means a corridor. That is, it is possible to calculate the probability of the CJP to move from one corridor to another. We express the conditional state probability by

$$\mathbb{P}_t((k,l),(a,b)) := \mathbb{P}\left(X_t^{(1)} \in (k,l) \,\middle|\, \mathcal{F}_{t-}, X_{t-}^{(1)} \in (a,b)\right),$$

for $(k,l) \in ((b,c),(c,d))$. We also define the interval

$$\mathcal{S}^{(k,l)} := \left[\frac{(k - X_{t-}^{(1)})}{\min\left(X_{t-}^{(1)} - a, d - X_{t-}^{(1)}\right)}, \frac{(l - X_{t-}^{(1)})}{\min\left(X_{t-}^{(1)} - a, d - X_{t-}^{(1)}\right)}\right],$$

for the denominator $\min(.) \neq 0$. Moreover, if $(\theta_t^{(1)})_{t\in\mathbb{T}}$ and $(J_t^{(1)})_{t\in\mathbb{T}}$ are mutually independent, then we have the following decomposition:

$$\mathbb{P}_t((k,l),(a,b)) = \mathbb{P}\left(\theta_t \in \mathcal{S}^{(k,l)} \,\middle|\, \mathcal{F}_{t-}, X_{t-}^{(1)} \in (a,b)\right) \mathbb{P}\left(\Delta J_t = 1 \mid \mathcal{F}_{t-}, X_{t-}^{(1)} \in (a,b)\right),$$

for $(k,l) \in ((b,c),(c,d))$. Since in this model $\theta_t \in [-1,1]/\{0\}$ for all $t \in \mathbb{T}$, it follows that $\mathbb{P}_t((k,l),(a,b)) = 0$ if $(k - X_{t-}^{(1)}) > \min(X_{t-}^{(1)} - a, d - X_{t-}^{(1)})$. This shows that $(X_t^{(1)})_{t\in\mathbb{T}}$ has a higher probability of moving to another corridor if it is closer to a boundary of that corridor. These probabilities can be calculated from any one corridor to another, i.e.,

$$\mathbb{P}_t((k,l),(c,d)) = \mathbb{P}\left(\theta_t \in \mathcal{S}^{(k,l)} \,\middle|\, \mathcal{F}_{t-}, X_{t-}^{(1)} \in (c,d)\right) \mathbb{P}\left(\Delta J_t = 1 \mid \mathcal{F}_{t-}, X_{t-}^{(1)} \in (c,d)\right),$$

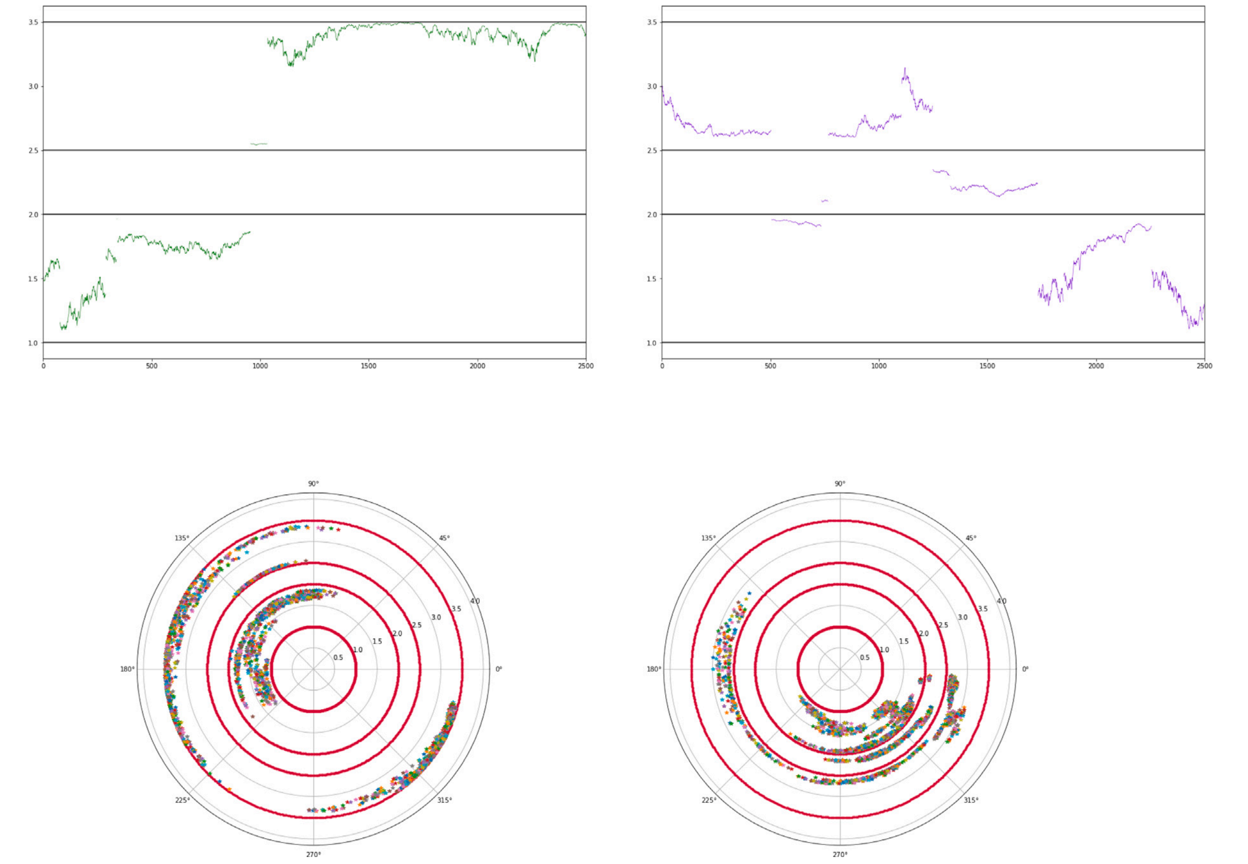

Fig. 4. Here, $a = 1,\ b = 2,\ c = 2.5,\ d = 3.5$. Also, $w_1 = w_2 = w_3 = 0.5$.

for $(k, l) \in ((a, b), (b, c))$. Since $\theta_t \in [-1, 1]/\{0\}$ for all $t \in \mathbb{T}$, we have $\mathbb{P}_t((k, l), (c, d)) = 0$ if $(l - X^{(1)}_{t-}) < -\min(X^{(1)}_{t-} - a, d - X^{(1)}_{t-})$. Finally, the probability of the CJP moving from the mid-corridor to either the inner or outer corridor is given by

$$\mathbb{P}_t((k, l), (b, c)) = \mathbb{P}\left(\theta_t \in \mathcal{S}^{(k,l)} \,\middle|\, \mathcal{F}_{t-}, X^{(1)}_{t-} \in (b, c)\right) \mathbb{P}\left(\Delta J_t = 1 \mid \mathcal{F}_{t-}, X^{(1)}_{t-} \in (b, c)\right),$$

for $(k, l) \in ((a, b), (c, d))$. Hence, the set $\mathcal{S}^{(k,l)}$ serves for all possible changes of state in this model. All expressions can be further simplified if $(\theta_t)_{t\in\mathbb{T}}$ and $(J_t)_{t\in\mathbb{T}}$ are mutually independent from all variables in the system such that

$$\mathbb{P}_t((k, l), (x_1, x_2)) = \mathbb{P}\left(\theta_t \in \mathcal{S}^{(k,l)} \,\middle|\, X^{(1)}_{t-} \in (x_1, x_2)\right) \mathbb{P}\left(\Delta J_t = 1\right).$$

This setup can be used to model a system in which a stochastic particle may jump from one energy state to another with a large-enough jump that is induced by a sufficiently strong exogenous shock, e.g., an energy pulse. For example, consider a quantum mechanical system, whereby one is interested in modelling the transition of the stochastic wave function of a quantum particle from one energy state to another. Another application is quantum tunnelling where the potential walls (barriers) of the quantum system that the particle "overcomes" could be modelled by the boundaries of an internal corridor, which are overcome by a large-enough jump. In this context, while the quantum particle is modelled by a captive jump process that can overcome walls, a classical particle would be modelled by a captive diffusion process trapped within a (possibly time-dependent) corridor, see [4]. The concept of domain boundaries (or walls), which produce clusters unable to disperse (if unaided by external intervention), abounds in many fields of physics, but it is also encountered in, e.g., finance (e.g., volatility clustering, herding in markets), chemistry, sociology, and psychology.

3.3. Attractors in dynamical systems

In dynamical systems, attractors represent (possibly multiple) physical states in a given environment to which the system closes in on. Using captive jump-processes, one can consider bounded stochastic systems that tend to any of these attractors as time passes, while providing the possibility that the system may jump from one attraction point (or region) to another, provided there is a strong enough force. We shall construct an example where we have two attracting points at time $t = T$. First, we set $\{\mathbb{T}^{(i)} = \mathbb{T}\}_{i=0,1,\dots,3}$.

As for the coordinates of the channels, we keep the outermost boundaries constant (for simplicity, as before), where $(g_t^{(0)})_{t\in\mathbb{T}} = a$, $(g_t^{(3)})_{t\in\mathbb{T}} = d$, and $-\infty < a < d < \infty$. The internal corridors are modelled as linear, time-dependent maps by

$$g_t^{(1)} = \begin{cases} \frac{a+d}{2} & t < t^*, \\ \frac{(a+d)}{2} - \frac{(d-a)}{2(T-t^*)}(t-t^*) & t \in [t^*, T], \end{cases}$$
$$g_t^{(2)} = \begin{cases} \frac{a+d}{2} & t < t^*, \\ \frac{(a+d)}{2} + \frac{(d-a)}{2(T-t^*)}(t-t^*) & t \in [t^*, T]. \end{cases}$$

where $t^* \in (t, T)$ is an arbitrary point in time after which the internal channels evolve in opposite directions. The internal boundaries produce a divide that begins at the mid-level $(a+d)/2$ of the overall confined space given by $[a, d]$. The boundary $(g_t^{(1)})_{t\in\mathbb{T}}$ converges linearly (downwards) to a as $t^* \le t \to T$, while $(g_t^{(2)})_{t\in\mathbb{T}}$ converges linearly (upwards) to d as $t^* \le t \to T$. Thus, we have

$$\frac{\mathrm{d}}{\mathrm{d}t} g_+^{(1)}(t) = -\frac{(d-a)}{2(T-t^*)} \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}t} g_+^{(2)}(t) = \frac{(d-a)}{2(T-t^*)},$$

for $t \ge t^*$. In this setting we can view a and d as attractors towards which we expect the system to close in on as $t \to T$. The dynamics of the considered captive jump process guarantee that the process either tends to attractor state a or attractor state d as $t \to T$, irrespective of the stochastic trajectory up to that time, or its starting point. In this setting, we propose the following structure for the captive jump process:

$$\mathrm{d}X_t = \kappa_t(\Psi_t)(\beta_t(\Psi_t) - X_t)\mathrm{d}t + \sigma(t, X_t; \boldsymbol{g}_t)\mathrm{d}W_t + \gamma(t-, \Psi_{t-}, X_{t-}; \boldsymbol{g}_{t-})\Delta J_t,$$

where $t \in [0, T)$ and $X_0 \in (a, g_0^{(1)})$ or $X_0 \in (g_0^{(2)}, d)$. Here, we have

$$\beta_t(\Psi_t) = \begin{cases} h_t^{(0)} & \text{if } X_{\tau_t} \in [a, g_t^{(1)}] \\ h_t^{(3)} & \text{if } X_{\tau_t} \in [g_t^{(2)}, d], \end{cases}$$

where $h^{(0)} : \mathbb{T}^{(1)} \to \mathbb{R}$ and $h^{(3)} : \mathbb{T}^{(2)} \to \mathbb{R}$ are continuous maps satisfying $a < h_t^{(0)} < g_t^{(1)}$ for every $t \in [0, T)$ and $g_t^{(2)} < h_t^{(3)} < d$ for every $t \in [0, T)$ such that $\lim_{t\to T} h_t^{(0)} = a$ and $\lim_{t\to T} h_t^{(3)} = d$, respectively. Moreover, $\{\kappa_t(\Psi_t)\}_{t\in[0,T)}$ is a process that satisfies

$$\kappa_t(\Psi_t) > \frac{(d-a)}{2(T-t^*)}(g_t^{(1)} - h_t^{(0)})^{-1} \quad \text{if } X_{\tau_t} \in [a, g_t^{(1)}] \text{ for } t \in [0, T),$$
$$\kappa_t(\Psi_t) > \frac{(d-a)}{2(T-t^*)}(h_t^{(3)} - g_t^{(2)})^{-1} \quad \text{if } X_{\tau_t} \in [g_t^{(2)}, d] \text{ for } t \in [0, T).$$

These requirements ensure that the drift function μ satisfies the conditions in Definition 2.21. We note that $\kappa_t(\Psi_t)$ is not defined at $t = T$, due to the singularity. As an example, if half of the spatial-distance between the outermost boundaries equals the time-distance between the divergence point t^* and T, with $(d-a)/2 = T - t^*$, we have

$$\kappa_t(\Psi_t) > \frac{1}{g_t^{(1)} - h_t^{(0)}} \quad \text{if } X_{\tau_t} \in [a, g_t^{(1)}] \text{ for } t \in [0, T),$$
$$\kappa_t(\Psi_t) > \frac{1}{h_t^{(3)} - g_t^{(2)}} \quad \text{if } X_{\tau_t} \in [g_t^{(2)}, d] \text{ for } t \in [0, T).$$

In addition, we set

$$\begin{aligned} \sigma(t, X_t; \boldsymbol{g}_t) &= \eta \prod_{j=1}^{4} (X_t - g_t^{(j)}), \\ &= \eta(X_t - a)(X_t - g_t^{(1)})(X_t - g_t^{(2)})(X_t - d), \end{aligned}$$

and

$$\gamma(t, \Psi_t, X_t; \boldsymbol{g}_t) = \begin{cases} w_t(d - X_t) + (1 - w_t)(g_t^{(2)} - X_t) & \text{if } X_{\tau_t} \in [a, g_t^{(1)}], \\ w_t(a - X_t) + (1 - w_t)(g_t^{(1)} - X_t) & \text{if } X_{\tau_t} \in [g_t^{(2)}, d], \end{cases}$$

where $w_t \in (0, 1)$ for every $t \in \mathbb{T}$. We now have a stochastic process that satisfies all the conditions in Definition 2.21, and so is a captive jump process. In this model, whenever there is a jump, the discontinuity takes $(X_t)_{t\in[0,T)}$ from one internal corridor to another. Thus, each jump necessarily provides a change of direction towards a different attractor from the most recent attractor's influence—this can of course be relaxed. We can control the likelihood of regime changes through the jump-time distribution of $(J_t)_{t\in\mathbb{T}}$. One may ask for $\mathbb{P}(\Delta J_t = 1) \to 0$ as $t \to T$, if one wanted to *decrease* the probability that the system jumps to another attractor state while it approaches the (current) attractor to which it is closest. Fig. 5 demonstrates the simulated behaviour of the aforementioned double-attractor system, where we choose

$$h_t^{(0)} = \frac{a + g_t^{(1)}}{2} \quad \text{and} \quad h_t^{(3)} = \frac{d + g_t^{(2)}}{2}.$$

Each path remains within the geometry as constructed above, where the system necessarily tends to one of the attractors located at a and d, as $t \to T$.

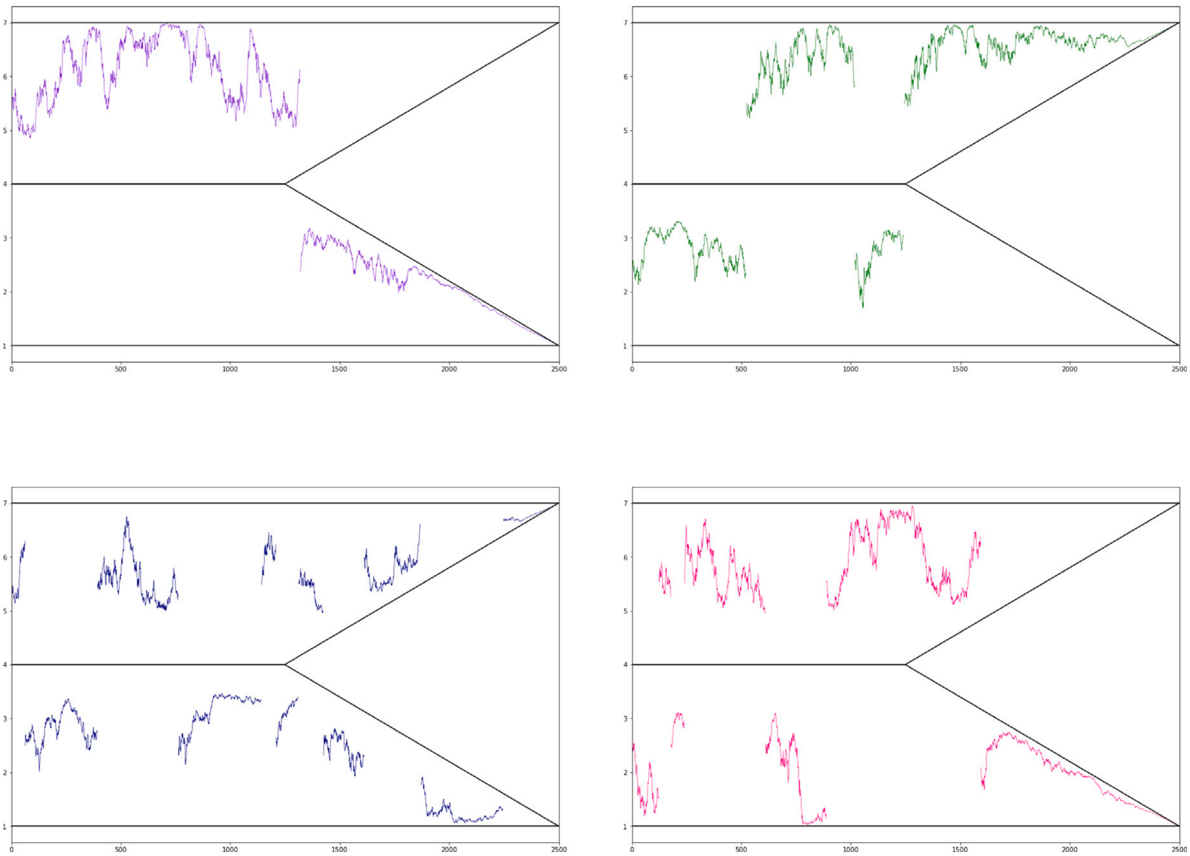

Fig. 5. Here, $a = 1$, $d = 7$, $t^* = T/2$, $\eta = 0.125$ and $w = 0.5$.

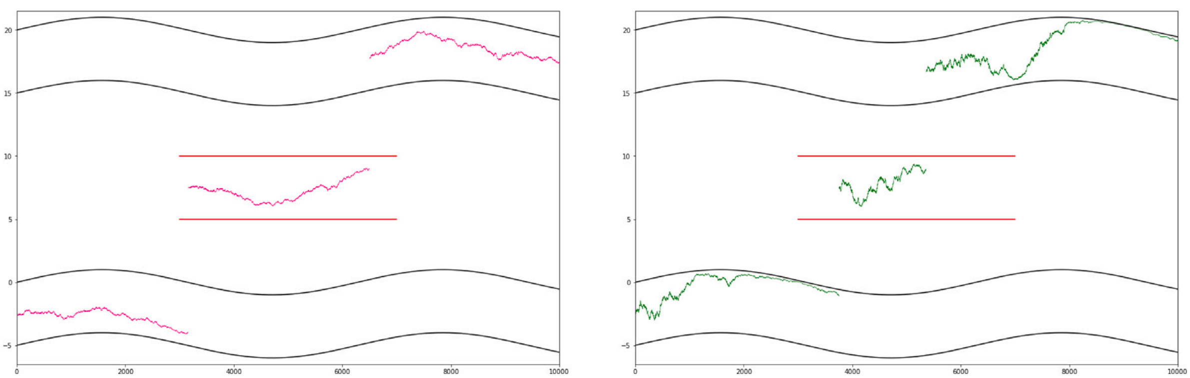

Fig. 6. Here, $a = -5$, $b = 0$, $c = 15$, $\eta = 0.3$ and $d = 20$.

3.4. Island visitors

We shall briefly demonstrate the dynamics of what we call an *island-visiting* captive jump process, where the geometry of the domain includes at least one isolated region the captive process jumps into, only to remain there temporarily. We shall clarify this description through a specific setting. We first draw attention to the simulated samples in Fig. 6 before detailing the mathematical model construction. The *island region* shown in Fig. 6 is the isolated domain between the sinusoidal corridors. In this example, we choose $m = 5$ such that $\mathbb{T}^{(0)} = \mathbb{T}^{(1)} = \mathbb{T}^{(4)} = \mathbb{T}^{(5)} = \mathbb{T}$ and $\mathbb{T}^{(2)} = \mathbb{T}^{(3)} \subset \mathbb{T}$. Here, $\mathbb{T}^{(2)}$ and $\mathbb{T}^{(3)}$ define the isolated region, where

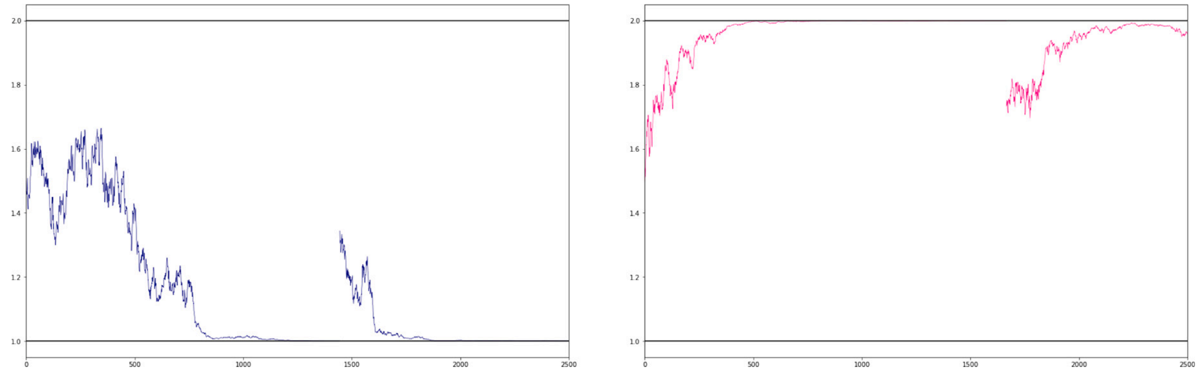

Fig. 7. Here, $a = 1$, $b = 2$ and $w_1 = w_2 = 0.5$.

$g_t^{(0)} < g_t^{(1)} < g_t^{(4)} < g_t^{(5)}$ for all $t \in \mathbb{T}$, and $g_t^{(1)} < g_t^{(2)} < g_t^{(3)} < g_t^{(4)}$ for all $t \in \mathbb{T}^{(2)} = \mathbb{T}^{(3)}$. In addition, for some $a < b < c < d$, we set $g_t^{(0)} = a+\sin(t)$, $g_t^{(1)} = b+\sin(t)$, $g_t^{(2)} = w_2 \max(g_t^{(1)} : t \in \mathbb{T})+(1-w_2)\min(g_t^{(4)} : t \in \mathbb{T})$, $g_t^{(3)} = w_3 \max(g_t^{(1)} : t \in \mathbb{T})+(1-w_3)\min(g_t^{(4)} : t \in \mathbb{T})$, $g_t^{(4)} = c + \sin(t)$, and $g_t^{(5)} = d + \sin(t)$, where $w_2, w_3 \in (0,1)$ such that $w_2 > w_3$. Due to the choice of the time-segments, we have $\tau_t^{(0)} = \tau_t^{(1)} = \tau_t^{(4)} = \tau_t^{(5)}$ and $\tau_t^{(2)} = \tau_t^{(3)}$. We let $X_0 \in (g_0^{(0)}, g_0^{(1)})$ and choose the drift function μ and volatility function σ of the CJP as follows:

$$\mu(t, \Psi_t, X_t, \boldsymbol{g}_t) = \begin{cases} \cos(t) & \text{if } X_{\tau_t^{(0)}} \in [g_t^{(0)}, g_t^{(1)}], \\ \frac{1}{2}(g_t^{(2)} + g_t^{(3)}) - X_t & \text{if } X_{\tau_t^{(2)}} \in [g_t^{(2)}, g_t^{(3)}] \text{ and } t \in \mathbb{T}^{(2)}, \\ \cos(t) & \text{if } X_{\tau_t^{(0)}} \in [g_t^{(4)}, g_t^{(5)}], \end{cases}$$

and

$$\sigma(t, \Psi_t, X_t, \boldsymbol{g}_t) = \begin{cases} \eta(X_t - g_t^{(0)})(X_t - g_t^{(1)}) & \text{if } X_{\tau_t^{(0)}} \in [g_t^{(0)}, g_t^{(1)}], \\ \eta(X_t - g_t^{(2)})(X_t - g_t^{(3)}) & \text{if } X_{\tau_t^{(2)}} \in [g_t^{(2)}, g_t^{(3)}] \text{ and } t \in \mathbb{T}^{(2)}, \\ \eta(X_t - g_t^{(4)})(X_t - g_t^{(5)}) & \text{if } X_{\tau_t^{(0)}} \in [g_t^{(4)}, g_t^{(5)}]. \end{cases}$$

As for the coefficient γ, we use

$$\gamma(t, \Psi_t, X_t; \boldsymbol{g}_t) = \begin{cases} \lambda(g_t^{(2)} - X_t) + (1-\lambda)(g_t^{(3)} - X_t) & \text{if } X_{\tau_t^{(0)}} \in [g_t^{(0)}, g_t^{(1)}] \text{ and } t \in \mathbb{T}^{(2)}, \\ \lambda(g_t^{(4)} - X_t) + (1-\lambda)(g_t^{(5)} - X_t) & \text{if } X_{\tau_t^{(2)}} \in [g_t^{(2)}, g_t^{(3)}] \text{ and } t \in \mathbb{T}^{(2)}, \\ 0 & \text{if } X_{\tau_t^{(0)}} \in [g_t^{(2)}, g_t^{(3)}], \end{cases}$$

where $\lambda \in (0,1)$. In this example, the CJP must jump from the region defined by $\{(g_t^{(0)}, g_t^{(1)})\}_{t\in\mathbb{T}^{(2)}}$ to the one determined by $\{(g_t^{(2)}, g_t^{(3)})\}_{t\in\mathbb{T}^{(2)}}$, and from $\{(g_t^{(2)}, g_t^{(3)})\}_{t\in\mathbb{T}^{(2)}}$ to $\{(g_t^{(4)}, g_t^{(5)})\}_{t\in\mathbb{T}^{(2)}}$. To ensure there are necessarily two jumps occurring over the time-segment $\mathbb{T}^{(2)} = \mathbb{T}^{(3)}$, we set $\mathbb{P}(J_{\max(\mathbb{T}^{(2)})} - J_{\min(\mathbb{T}^{(2)})} = 2) = 1$. Hence, the captive process visits the isolated region located between the sinusoidal corridors almost surely, but only temporarily.

3.5. Captive jump processes within adhesive boundaries

In various applications, one observes adhesive behaviour whereby a stochastic process is temporarily absorbed at a boundary (if it hits that boundary) until the process leaves the boundary again after a finite amount of time. For example, in biology, molecules may evolve near sticky cell membranes, see [41], in epidemics, pathogens can behave in an adhesive way at zero-concentration levels, see [42].

CJPs can contribute to the mathematical foundations of such literature. As an example, we model situations where the process spends some time on the boundary it has hit, until a jump occurs allowing it to bounce back into the internal domain. We simulate the following CJP:

$$\mathrm{d}X_t = \sin(X_t - a)\sin(X_t - b)\mathrm{d}W_t + \theta_{t-}\left(X_{t-} - (w_1 a + w_2 b)\right)\Delta J_t, \tag{12}$$

for $X_0 \in (a,b)$, and where $\theta_t \in [-1,0)$ for all $t \in \mathbb{T}$, $w_1 \in (0,1)$ and $w_2 = 1 - w_1$.

In Fig. 7, the sample paths show that the process is absorbed at a boundary whenever it hits that boundary and stays there until a jump occurs that takes it back into the internal domain. We highlight that the dynamics (12) are just one example amongst many that belong to a large family of captive jump processes featuring adhesive behaviour.

Size of Mesh (m)	Discretisation (δ = 1/m)	Constant Boundary (Application 3.2)			Branching Boundary (Application 3.3)			Periodic Boundary (Application 3.4)		
		Metric 1	*Metric 2*	*Metric 3*	*Metric 1*	*Metric 2*	*Metric 3*	*Metric 1*	*Metric 2*	*Metric 3*
100	0.0100	0.010%	3.437%	9.300%	0.047%	12.970%	35.800%	0.023%	0.001%	22.700%
250	0.0040	0.000%	0.000%	0.000%	0.002%	6.022%	5.500%	0.007%	0.000%	17.000%
500	0.0020	0.000%	0.000%	0.000%	0.000%	1.598%	0.300%	0.002%	0.000%	11.800%
1000	0.0010	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.001%	0.000%	7.700%
2500	0.0004	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	4.900%
5000	0.0002	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	2.500%

Fig. 8. Numerical performance metrics.

3.6. Numerical sensitivity analysis

As mentioned above, since CJPs in general do not admit closed-form representations and may evolve in highly complex geometries, simulating CJPs becomes soon non-trivial, where standard Euler–Maruyama schemes could fail to keep all CJP paths within their bounded domains. As an example, we refer to [43] and the references therein for the complex nature of simulating Wright-Fisher diffusions, which form a specific example of CJPs that exclude jumps within constant boundaries. Studying exact simulations of CJPs is a significantly more involved proposition, and it goes beyond the scope of this paper. The development of a general simulation scheme for CJPs deserves separate and dedicated research. Nonetheless, we shall hereby quantify *numerical violations* produced by the Euler–Maruyama scheme when simulating CJPs to show how such violations diminish as the time-step δ becomes shorter—and so the theoretical setting is approached. Later, we present an implicit method to remove all violations for practical applications. The analysis of the distributional convergence properties of the proposed new numerical method is left for future research.

The sensitivity table presents three applications detailed above in order to demonstrate how numerical violations behave across different geometries and choices of δ. We identify three *violation metrics*, of which two are *tolerance measures* and the third is an *absolute measure*. The tolerance measures express "how badly" boundaries have been exceeded: the first such measure tells one how often the violation has occurred, while the second expresses a severity level in terms of by how much a boundary has been exceeded. The third metric, i.e., the absolute measure, states how many of the process trajectories (i.e., sample paths) violate a boundary, no matter how often or by how much. Hence, the third metric can be regarded as the most punitive among the three. Here are the *violation metrics* utilized in our analysis:

- Metric 1, temporal tolerance measure for violation ("how often"):

$$\frac{\text{number of violated time points}}{(\text{total number of paths}) \times (\text{total number of time points})}$$

- Metric 2, spatial tolerance measure for violation ("how much"):

$$\frac{\text{maximum violated distance}}{\text{local tunnel width}}$$

- Metric 3, absolute measure for violation ("how many"):

$$\frac{\text{number of violated paths}}{\text{total number of paths}}$$

The metrics above quantify different aspects of what we collectively call numerical violations. *Metric 1* is the ratio of the number of time points when the simulated values reside outside of the expected domain with respect to the total number of simulated paths multiplied by the total number of time points in the mesh. *Metric 2* quantifies the magnitude (i.e., the severity) of the violations in terms of the maximum distance that the simulated values take outside of the expected boundaries (i.e., tunnels) at any time t_k. *Metric 3* is the number of simulated paths which fail to be CJPs because they escape their boundaries at some t_k during their lifetime (see Fig. 8).

All values in the sensitivity analysis are based on 1000 simulated paths for each cell of the table. It can be seen that each metric resolves differently for each application being considered, since each CJP model has a different level of complexity. For example, the constant-boundary setup of Example 3.2 not only starts with lower violation metrics, but also converges to zero violation the quickest as δ decreases. On the other hand, Example 3.2 and Example 3.3 converge at different speeds with respect to different metrics. While *Metric 3* of Example 3.2 starts off with a higher degree of violation than that of Example 3.3, Example 3.2 gives evidence for a faster convergence when compared to Example 3.3 as δ decreases. We highlight the convergence dynamics of *Metric 3* below (see Fig. 9).

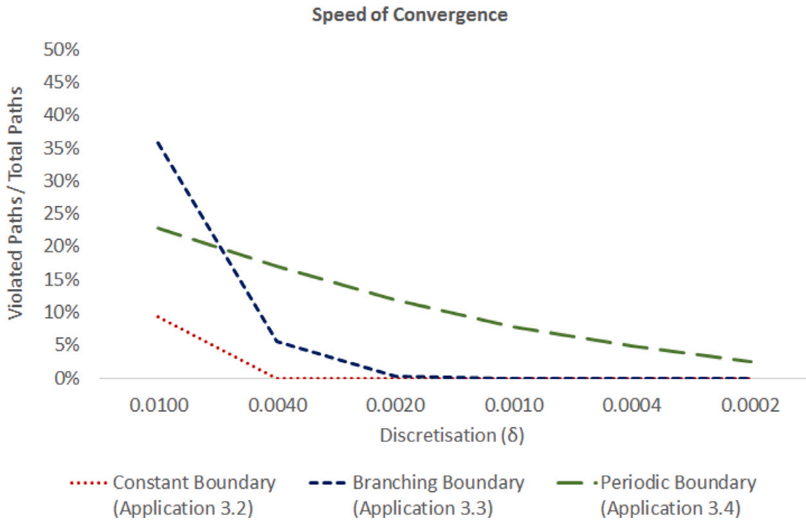

Fig. 9. Speed of convergence for *Metric 3*.

In summary, the sensitivity analysis demonstrates how numerical violations account for decreasing values of the aforementioned systems as the mesh becomes finer, but also highlights how one needs to be careful when simulating CJPs in practice by using the classical Euler–Maruyama scheme.

We now recall $(\hat{X}_{t_k})_{t_k\in\mathbb{T}}$ is an approximation of $(X_t)_{t\in\mathbb{T}}$ over the discretized grid $0 = t_0 \le t_1 \le \cdots \le t_m \le= T < \infty$ for some $m \in \mathbb{N}_+$, and generate the implicit scheme that adjusts every violation by projecting their value onto the closest boundary that is violated at any t_k. In doing so, we define $\mathcal{D}_{t_k}$ as the set of all possible values determined by the boundaries $\mathbf{g}_{t_k}$ that X_{t_k} is *expected* to take at t_k. In other words, $\mathcal{D}_{t_k}$ is the theoretical domain of X_{t_k} as proven in Proposition 2.22. In addition, we define $g^{(*)}_{t_k}$ as the boundary that is violated at t_k when $X_{t_k} \notin \mathcal{D}_{t_k}$. The adjusted algorithm for the implicit Euler–Maruyama scheme we propose is as follows:

Adjusted implicit algorithm

1. For $t = t_0$, set $\hat{X}_t = x_0$ and $\Delta J_t = 0$.
2. For $t = t_k$, let
   $\hat{X}_{t_{k+1}} = \hat{X}_t \; + \; \mu\left(t, \Psi_t, \hat{X}_t; g_t\right)\delta \; + \; \sigma\left(t, \Psi_t, \hat{X}_t; g_t\right)\left(W_{t_{k+1}} - W_t\right) + A_t$, where
   $A_t = \gamma\left(t, \Psi_t, \hat{X}_t; g_t\right)$ if $\Delta J_t = 1$, otherwise $A_t = 0$.
3. If $\hat{X}_{t_{k+1}} \notin \mathcal{D}_{t_{k+1}}$, set $\hat{X}_{t_{k+1}} = g^{(*)}_{t_{k+1}}$.
4. Repeat Step 2 and 3 for $k = 1, \ldots, m-1$,

where $\delta = T/m$, $t_k = k\delta$, and $\Delta J_{t_k} \in \{0, 1\}$ for every $t_k \in \mathbb{T}$.

This adjusted algorithm guarantees that all simulated process trajectories stay within the constraining boundaries. As such, Metric 3 will return zero violating paths, which of course also makes Metric 1 and 2 yield a zero value.

Next, we apply the adjusted implicit algorithm above to simulate 1000 paths of a CJP—none of the paths will exceed the given boundaries (see Figs. 10 and 11).

By construction of the scheme above, there are zero violations for any choice of δ due to the adjustment Step 3. We shall call the discrete-time constrained stochastic process $(X_{t_k})_{t_k\in\mathbb{T}}$ generated by the proposed implicit algorithm the *empirical captive jump process*, of which formal numerical analysis (e.g., distributional convergence properties) we leave for future research.

4. Endogenous CJPs

We briefly discuss how a specific family of captive jump processes arises naturally in nonlinear filtering when so-called piecewise-enlarged filtrations—as introduced in [44] for energy-based quantum state reduction—are used to model noisy information flows. To avoid distraction from the core topic of this paper, we refer the reader to [44] for details on the piecewise-enlarged filtration framework, and instead present here only the necessary parts for our purposes.

4.1. Endogenous captive jump processes in nonlinear filtering

Let $(\tau_i)_{i=1}^n$ be an n-sequence of $(\mathcal{F}_t)$-stopping times such that $0 < \tau_1 < \tau_2 < \cdots < \tau_n < \infty$. We associate the sequence of stopping times to càdlàg Heaviside processes $(H_{\tau_i}(t))_{t\in\mathbb{T}}$ via

$$H_{\tau_i}(t) = \begin{cases} 1 & \text{if } \tau_i \le t, \\ 0 & \text{otherwise,} \end{cases}$$

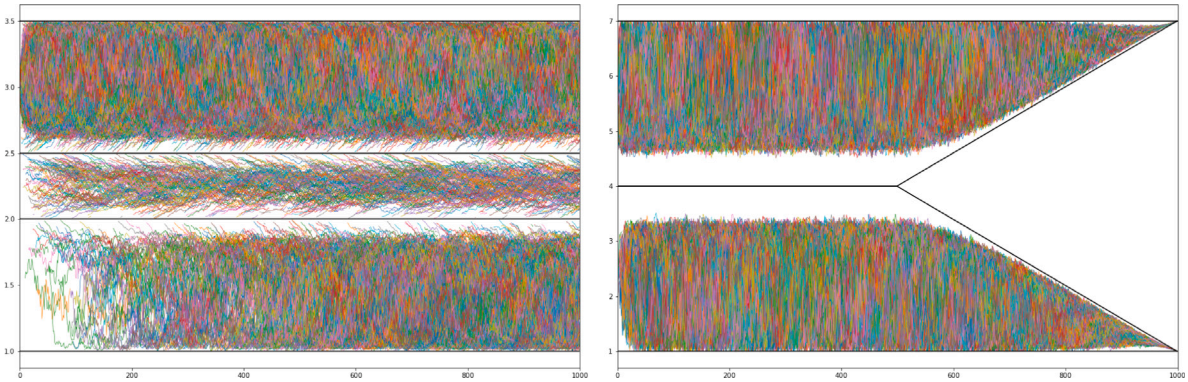

Fig. 10. Left: Application 3.2. Right: Application 3.3.

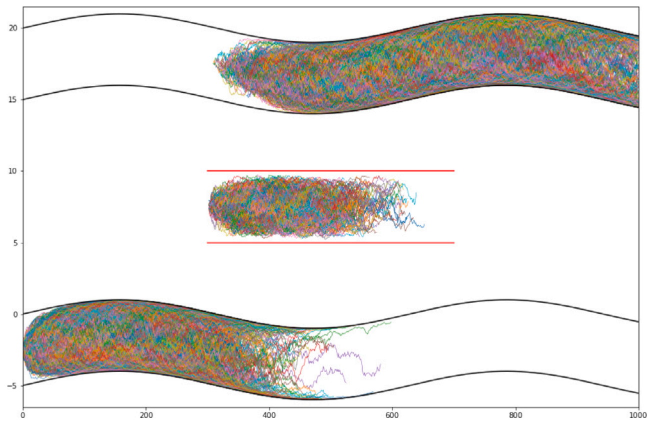

Fig. 11. Application 3.4.

for $i = 1, \dots, n$. We let $(\xi_t^{(i)})_{t\in\mathbb{T}}$ denote a noisy information process, mutually independent of each τ_i, given by $\xi_t^{(i)} = (t/T)Z + B_{tT}^{(i)}$, where each $(B_{tT}^{(i)})_{t\in\mathbb{T}}$ is a standard Brownian bridge that is independent of the square-integrable random variable Z, for $i = 1, \dots, n+1$. This type of information process and construction of filtering systems were introduced in the information-based asset pricing approach, see [45,46] for early works, and [47] for a collection of publications in this field. Using the aforementioned processes, we construct a piecewise-enlarged filtration $(\mathcal{G}_t)_{t\in\mathbb{T}}$, where $\mathcal{G}_t$ is a sub-algebra of $\mathcal{F}_t$ for all $t \in \mathbb{T}$, as follows:

$$\mathcal{G}_t = \sigma((\xi_s^{(1)})_{0\le s\le t}) \bigvee \left(\bigvee_{i=1}^{n} \mathcal{V}_t^{\xi^{(i+1)}} \right),$$

where each $(\mathcal{V}_t^{\xi^{(i)}})_{t\in\mathbb{T}}$ is given by

$$\mathcal{V}_t^{\xi^{(i+1)}} = \begin{cases} \sigma((H_{\tau_i}(s))_{0\le s\le t}) & \tau_i > t, \\ \sigma((H_{\tau_i}(s))_{0\le s\le t}, (\xi_s^{(i+1)})_{\tau_i\le s\le t}) & \tau_i \le t, \end{cases}$$

for $i = 1, \dots, n$ and $t \in \mathbb{T}$. We now define the conditional expectation process $(X_t)_{t\in\mathbb{T}}$ as our $\mathcal{L}^2$-best-estimate of Z given $(\mathcal{G}_t)_{t\in\mathbb{T}}$ by

$$X_t = \mathbb{E}[Z \mid \mathcal{G}_t]. \tag{13}$$

In [44] it is shown that $X_t = \mathbf{X}_t^\top \mathbf{I}_t$ for all $t \in \mathbb{T}$, where

$$\mathbf{X}_t = \begin{bmatrix} \mathbb{E}\left[Z \,\middle|\, \xi_t^1\right] \\ \vdots \\ \mathbb{E}\left[Z \,\middle|\, \hat{\xi}_t^{(j)}\right] \\ \vdots \\ \mathbb{E}\left[Z \,\middle|\, \hat{\xi}_t^{(n+1)}\right] \end{bmatrix} \quad \text{and} \quad \mathbf{I}_t = \begin{bmatrix} 1 - H_{\tau_1}(t) \\ \vdots \\ H_{\tau_{j-1}}(t)(1 - H_{\tau_j}(t)) \\ \vdots \\ H_{\tau_n}(t) \end{bmatrix},$$

given that each so-called effective information process $(\hat{\xi}_t^{(j)})_{t\in\mathbb{T}}$ can be represented by

$$\hat{\xi}_t^{(j)} = X\frac{\sqrt{j}t}{T} + \hat{B}_{tT}^{(i)},$$

with $(\hat{B}_{tT}^{(i)})_{t\in\mathbb{T}}$ a standard Brownian bridge. The process $(X_t)_{t\in\mathbb{T}}$ satisfies the SDE

$$\begin{aligned} X_t = \mathbb{E}[Z] &+ \sum_{j=1}^{n+1}\int_0^t \frac{\sqrt{j}}{(T-s)}\left(\mathrm{Var}\left[Z\,\middle|\,\hat{\xi}_s^{(j)}\right]\right) I_s^{(j)}\,\mathrm{d}W_s^{(j)} \\ &+ \sum_{j=2}^{n+1}\int_0^t \left(\mathbb{E}\left[Z\,\middle|\,\hat{\xi}_s^{(j)}\right] - \mathbb{E}\left[Z\,\middle|\,\hat{\xi}_s^{(j-1)}\right]\right)\delta_{\tau_{j-1}}(\mathrm{d}s), \end{aligned} \tag{14}$$

for $t\in\mathbb{T}$, where each $(W_t^{(j)})_{t\in\mathbb{T}}$ is a standard Brownian motion and $I_t^{(j)}$ is the jth element of $\mathbf{I}_t$. Now we choose $Z:\Omega\to\{k_l,k_u\}$, such that

$$0<\mathbb{P}(Z=k_l)<1 \quad\text{and}\quad \mathbb{P}(Z=k_u)=1-\mathbb{P}(Z=k_l).$$

As discussed in [48], it can be shown that

$$\mathbb{P}(Z=k_l\mid\hat{\xi}_t^{(j)}) = \left(1+\exp\left(-\frac{1}{2}\frac{\sqrt{j}(k_u-k_l)}{(T-t)}\left(\frac{t}{T}(k_l+k_u)-2\hat{\xi}_t^{(j)}\right)\right)\frac{\mathbb{P}(Z=k_u)}{\mathbb{P}(Z=k_l)}\right)^{-1},$$

$$\mathbb{P}(Z=k_u\mid\hat{\xi}_t^{(j)}) = \left(1+\exp\left(\frac{1}{2}\frac{\sqrt{j}(k_u-k_l)}{(T-t)}\left(\frac{t}{T}(k_l+k_u)-2\hat{\xi}_t^{(j)}\right)\right)\frac{\mathbb{P}(Z=k_l)}{\mathbb{P}(Z=k_u)}\right)^{-1},$$

from which it follows that

$$\begin{aligned} \mathrm{Var}\left[Z\,\middle|\,\hat{\xi}_s^{(j)}\right] &= \left(\mathbb{E}\left[Z\,\middle|\,\hat{\xi}_s^{(j)}\right]-k_l\right)\left(k_u-\mathbb{E}\left[Z\,\middle|\,\hat{\xi}_s^{(j)}\right]\right) \\ &= \left(X_t^{(j)}-k_l\right)\left(k_u-X_t^{(j)}\right), \end{aligned}$$

where $X_t^{(j)}$ is the jth element of $\mathbf{X}_t$. Therefore, with $x_0=(k_l\mathbb{P}(Z=k_l)+k_u\mathbb{P}(Z=k_u))\in(k_l,k_u)$ and the orthogonality of each element of $\mathbf{I}_t$, Eq. (14) takes the form

$$\begin{aligned} X_t &= x_0 + \sum_{j=1}^{n+1}\int_0^t \frac{\sqrt{j}}{(T-s)}\left(X_t^{(j)}-k_l\right)\left(k_u-X_t^{(j)}\right) I_s^{(j)}\,\mathrm{d}W_s^{(j)} + \sum_{j=2}^{n+1}\int_0^t\left(X_t^{(j)}-X_t^{(j-1)}\right)\delta_{\tau_{j-1}}(\mathrm{d}s), \\ &\overset{\text{law}}{=} x_0 + \int_0^t \frac{C_s}{(T-s)}\left(X_t-k_l\right)\left(k_u-X_t\right)\mathrm{d}W_s + \sum_{0\le s\le t}\left(X_s-X_{s-}\right)\Delta J_s, \end{aligned} \tag{15}$$

where $(W_t)_{t\in\mathbb{T}}$ is a standard Brownian motion, and $(J_t)_{t\in\mathbb{T}}$ and $(C_t)_{t\in\mathbb{T}}$ are adapted to $(\mathcal{G}_t)_{t\in\mathbb{T}}$ and are given by

$$J_t = \sum_{0\le s\le t}\sum_{i=1}^{n} H_{\tau_i}(s) \quad\text{and}\quad C_t = \left(1+\sum_{0<s\le t}\mathbb{1}(\Delta X_s\neq 0)\right)^{\frac{1}{2}},$$

respectively, for every $t\in\mathbb{T}$. In [49], Proposition 2.9, it is shown that there exists a random variable Y and a function h such that

$$h(Y)-X_{t-}\overset{\text{law}}{=}X_t-X_{t-}.$$

Thus, Eq. (15) can be expressed more succinctly through

$$X_t \overset{\text{law}}{=} x_0 + \int_0^t \frac{C_s}{(T-s)}\left(X_t-k_l\right)\left(k_u-X_t\right)\mathrm{d}W_s + \sum_{0\le s\le t}\gamma\left(s-,X_{s-};k_l,k_u\right)\Delta J_s,$$

which satisfies all the conditions given in Definition 2.21. We emphasize that $(C_t)_{t\in\mathbb{T}}$ can be constructed via the monitoring process $(\Psi_t)_{t\in\mathbb{T}}$. Hence, there exists a captive jump process that is equal in law to the $\mathcal{L}^2$-best-estimate of Z given by Eq. (13). Accordingly, in this specific setting with $Z:\Omega\to\{k_l,k_u\}$, where $0<\mathbb{P}(Z=k_l)<1$ and $\mathbb{P}(Z=k_u)=1-\mathbb{P}(Z=k_l)$, the captive jump process allows one to represent the superposition expression in Eq. (14) in a more parsimonious way.

4.2. Interacting captive jump processes

We next present a multivariate extension of the aforementioned framework that allows CJPs to interact with each other while maintaining their captive properties. Let $(\mathbf{X}_t)_{t\in\mathbb{T}}$ be a multidimensional captive jump process given by

$$\mathbf{X}_t = [X_t^{(1)},\dots,X_t^{(i)},\dots,X_t^{(n)}]^\top,$$

for $n\in\mathbb{N}_+$, where $(X_t^i)_{t\in\mathbb{T}}$ is CJP defined by Definition 2.2, for all $i\in\mathbb{N}_+$. To identify a value for the ith coordinate of $(\mathbf{X}_t)$, we define

$$\mathbf{X}_t[i;x] \triangleq [X_t^{(1)},\dots,x,\dots,X_t^{(n)}],$$

for any $i = 1, \dots, n$. We introduce $\mathbb{T}^{(i,j)} \subseteq \mathbb{T}$ for $j = 0, \dots, m^{(i)}$ and $i = 1, \dots, n$, where $m^{(i)}$ depends on index i since the number of time-segments can be different for each coordinate of $(\mathbf{X}_t)_{t\in\mathbb{T}}$. Here, we have

$$\mathbb{T}^{(i,j)} = [\tau^{(i,j)}_{\text{start}}, \tau^{(i,j)}_{\text{end}}] \quad \text{with} \quad 0 \le \tau^{(i,j)}_{\text{start}} < \tau^{(i,j)}_{\text{end}} \le T,$$

where $\mathbb{T}^{(i,0)} = \mathbb{T}^{(i,m^{(i)})} = \mathbb{T}$ for all $i = 1, \dots, n$. Over each time-segment, there is a boundary function that is collected in

$$\mathbf{G}_t = \{ g^{(i,j)}_t \,:\, j = 0, \dots, m^{(i)} \ \text{ for } \ i = 1, \dots, n \}$$

where $g^{(i,j)} \in \mathcal{G}(\mathbb{R})$ for every i, j. We now introduce progressively-measurable and increasing processes $(\tau^{(i,j)}_t)_{t\in\mathbb{T}^{(i,j)}}$ given by

$$\tau^{(i,j)}_t = \tau^{(i,j)}_{\text{start}} \vee \sup(s \,:\, \Delta X^{(i)}_s \neq 0 \quad \text{for} \quad \tau^{(i,j)}_{\text{start}} \le s \le t \in \mathbb{T}^{(i,j)}), \tag{16}$$

for $j = 0, \dots, m^{(i)}$, where $\sup \emptyset = -\infty$. Hence, if there is no jump in a given time period $\mathbb{T}^{(i,j)}$, then $\tau^{(i,j)}_t = \tau^{(i,j)}_{\text{start}}$ for all t in the period $\mathbb{T}^{(i,j)}$. We now define non-anticipate set-valued processes

$$\Psi^{(i)}_t = \left\{ X^{(i)}_{\tau^{(i,j)}_t} \,:\, \tau^{(i,j)}_t \le t, \ \text{for } j = 0, \dots, m^{(i)} \right\},$$

for $i = 1, \dots, n$ that are collected in the monitoring process $(\boldsymbol{\Psi}_t)_{t\in\mathbb{T}}$ as follows:

$$\boldsymbol{\Psi}_t = \bigcup_{i=1}^{n} \Psi^{(i)}_t.$$

Measurability and continuity for $(\boldsymbol{\Psi}_t)$ should be understood as presented in Section 2. Accordingly, a multivariate, internally piecewise-confined captive jump process $(\mathbf{X}_t)_{t\in\mathbb{T}}$ is governed by the system

$$X^{(i)}_t = x^{(i)}_0 + \int_0^t \mu^{(i)}\left(s, \boldsymbol{\Psi}_s, \boldsymbol{X}_s; \mathbf{G}_s\right) \mathrm{d}s + \int_0^t \sigma^{(i)}\left(s, \boldsymbol{\Psi}_s, \boldsymbol{X}_s; \mathbf{G}_s\right) \mathrm{d}M^{(i)}_s + \sum_{0\le s\le t} \gamma^{(i)}\left(s-, \boldsymbol{\Psi}_{s-}, \boldsymbol{X}_{s-}; \mathbf{G}_{s-}\right) \Delta J^{(i)}_s,$$

where $X^{(i)}_0 = x^{(i)}_0 \in [g^{(i,0)}_0, g^{(i,m^{(i)})}_0)$ for $i = 1, \dots, n$. The maps $\mu^{(i)}$ and $\sigma^{(i)}$ are continuous (possibly except at points where $\Delta J^{(i)} \neq 0$ with bounded jumps), and $\gamma^{(i)}$ is a locally bounded càdlàg map such that

1. $\mu^{(i)}\left(t, \boldsymbol{\Psi}_t, \mathbf{X}_t[i; g^{(i,j)}_t]; \mathbf{G}_t\right) \ge \mathrm{d}g^{(i,j)}_+(t)/\mathrm{d}t$ if $X^{(i)}_{\tau^{(i,j)}_t} \ge g^{(i,j)}_{\tau^{(i,j)}_t}$, for any $t \in \mathbb{T}^{(i,j)}$ where $X^{(i)}_t = g^{(i,j)}_t$,
2. $\mu^{(i)}\left(t, \boldsymbol{\Psi}_t, \mathbf{X}_t[i; g^{(i,j)}_t]; \mathbf{G}_t\right) \le \mathrm{d}g^{(i,j)}_+(t)/\mathrm{d}t$ if $X^{(i)}_{\tau^{(i,j)}_t} < g^{(i,j)}_{\tau^{(i,j)}_t}$, for any $t \in \mathbb{T}^{(i,j)}$ where $X^{(i)}_t = g^{(i,j)}_t$,
3. $\sigma^{(i)}\left(t, \boldsymbol{\Psi}_t, \mathbf{X}_t[i; g^{(i,j)}_t]; \mathbf{G}_t\right) = 0$, for any $t \in \mathbb{T}^{(i,j)}$ where $X^{(i)}_t = g^{(i,j)}_t$,
4. $g^{(i,0)}_{t-} - X^{(i)}_{t-} \le \gamma^{(i)}\left(t-, \boldsymbol{\Psi}_{t-}, \boldsymbol{X}_{t-}; \mathbf{G}_{t-}\right) \le g^{(i,m)}_{t-} - X^{(i)}_{t-}$ for all $t \in \mathbb{T}$,

for $j = 0, \dots, m^{(i)}$ and $i = 1, \dots, n$, $\mathbb{P}$-a.s., given that $(\boldsymbol{M}_t)_{t\in\mathbb{T}} \in \mathcal{M}(\mathbb{R}^n)$ and $(\boldsymbol{J}_t)_{t\in\mathbb{T}} \in \mathcal{J}(\mathbb{R}^n)$ are mutually independent.

Proposition 4.1. *The following statements hold $\mathbb{P}$-almost-surely.*

1. *For any $t \in \mathbb{T}$, $g^{(i,0)}_t \le X^{(i)}_t \le g^{(i,m^{(i)})}_t$.*
2. *For any $j \neq 0$ and $j \neq m^{(i)}$, if $X^{(i)}_{\tau^{(i,j)}_{start}} \ge g^{(i,j)}_{\tau^{(i,j)}_{start}}$ and $\Delta J^{(i)}_t = 0$ for all $t \in \mathbb{T}^{(i,j)}$, then $X^{(i)}_t \ge g^{(i,j)}_t$ for all $t \in \mathbb{T}^{(i,j)}$. If $\Delta J^{(i)}_t = 1$ for some $t \in \mathbb{T}^{(i,j)}$ then $\mathbb{P}(X^{(i)}_t < g^{(i,j)}_t) \ge 0$ for $t \in \mathbb{T}^{(i,j)}$.*
3. *For any $j \neq 0$ and $j \neq m^{(i)}$, if $X^{(i)}_{\tau^{(i,j)}_{start}} < g^{(i,j)}_{\tau^{(i,j)}_{start}}$ and if $\Delta J^{(i)}_t = 0$ for all $t \in \mathbb{T}^{(i,j)}/(\tau^{(i,j)}_{start})$, then $X^{(i)}_t \le g^{(i,j)}_t$ for all $t \in \mathbb{T}^{(i,j)}/(\tau^{(i,j)}_{start})$. If $\Delta J^{(i)}_t = 1$ for some $t \in \mathbb{T}^{(i,j)}$ then $\mathbb{P}(X^{(i)}_t > g^{(i,j)}_t) \ge 0$ for $t \in \mathbb{T}^{(i,j)}/(\tau^{(i,j)}_{start})$.*

We omit the proof of Proposition 4.1 since it follows similar steps as those in the proof of Proposition 2.22. The multivariate setup allows for the coordinates of $(\mathbf{X}_t)_{t\in\mathbb{T}}$ to display interacting captive dynamics with discontinuities within bounded domains.

5. Conclusions

Captive jump processes (CJPs) are constrained stochastic processes that inherently embed random discontinues in their paths, and cannot escape their pre-specified confined space. The flexibility of the proposed mathematical framework can be formulated by controlling the process trajectories via deterministic time-dependent boundary functions that form the constrained space. While these boundary functions control the drift, volatility and jump size processes, the random jump times and the driving diffusion process can be modelled independently. The modelling richness afforded by CJPs, and the mechanism by which the controlled jumps are modulated in terms of the boundary functions, motivates its own mathematical development and justifies the dedicated study in this paper. We anticipate many applications in fields across the natural, life and social sciences, and provide several explicit examples: (i) CJPs evolving within circular domains suggesting use in quantum physics and chemistry, (ii) CJPs in a dynamical system with attraction regions suggesting use in systems with gravitational tunnels from which the process can only escape with a jump of suitable size, and (iii) CJPs in biological ecosystems in which adhesive agents are being absorbed at a boundary for some time before a jump gives the process a new lease of stochastic life within the confined space. Whenever a phenomenon requires the modelling of constrained stochastic jump dynamics that may or may not involve internal corridors, the proposed family of CJPs can be considered as an alternative candidate that finds tractable applications via numerical simulations.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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