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On Optimal Toll Design for Bosporus Crossings

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Boğaziçi Geçişleri için Optimal Geçiş Ücreti Tasarımı Üzerine

Abstract

For many years, two toll bridges served commuter demand to cross the strait called Bosporus in Istanbul, Turkey. An underground connection called the Eurasian tunnel had been recently launched to relieve the strait's traffic. We study a simple transportation model that incorporates the forces that have come into play after the opening of the Eurasian tunnel. We find that for welfare maximisation, the premium paid for using the tunnel should be fixed in the two directions and not excessive. The current toll regime violates these features, and we recommend its amendment in light of our findings.

Keywords : Congestion, Toll Bridges, Mixed Duopoly, Regulation, Bosporus.

JEL Classification Codes : C72, L13, L32, L38, L51, R40, R48.

Öz

Uzun yıllar boyunca, İstanbul Boğazı’nı geçmek isteyen taşıtlara iki ücretli köprü hizmet verdi. Yakın bir zaman önce ise, Boğaz’da oluşan trafiği rahatlatmak amacı ile Avrasya tüneli adı verilen bir yeraltı bağlantısı hizmete açıldı. Bu çalışmada, Avrasya tünelinin açılmasından sonra devreye giren güçleri bünyesinde barındıran basit bir ulaşım modelini ele aldık. Yaptığımız analizler, refahın maksimizasyonu için, tünel kullanımı için ödenen primin iki yönde aynı olması ve aşırı olmaması gerektiğini ortaya koyuyor. Mevcut geçiş ücret rejimi bu özellikleri ihlal etmekte. Bulgularımızın ışığında mevcut geçiş ücret rejiminde iyileştirme yapılmasını öneriyoruz.

Anahtar Sözcükler : Trafik Sıkışıklığı, Ücretli Köprü, Yarı Kamusal Düopol, Regülasyon, Boğaziçi.

1. Introduction

Traffic congestion is a big problem in Istanbul, Turkey. According to TomTom's annually released Traffic Index, Istanbul was the 5th most congested city in the world in 2020 and the most congested one in 2015 (among over 400 cities). In a recent survey study by Aydın et al. (2019), Istanbulites ranked traffic congestion as the gravest problem of the town, above unemployment, urban transformation, and the shortcomings in education and urban infrastructure. The city's annual traffic congestion cost is around £2 billion.

The city's most congested traffic is probably over the Bosporus, a narrow strait that divides the city into its Asian and European sides. Of the city's 15 million residents, about one-third live on the Asian and two-thirds on the European side. As many residents live and work on two different sides, there is a high daily commuter demand to cross the strait. Until 2016, this demand had been served by two bridges. To relieve the strait's traffic, an underground connection was launched in December 2016, called the Eurasian tunnel, located in the densely populated south of the city[1]. But the strait's traffic problem is far from over, especially in rush hours. As a reference for comparison, the average daily number of vehicles using the two bridges in both directions was 386,400 in 2015 and 319,710 in 2018; for details, see (Kara Yolları Genel Müdürlüğü, 2019; 2016).

In this paper, we study optimal toll design for Bosporus crossings. Tolling vehicles on congested roads is a common practice around the world. It makes economic sense since a vehicle's use of a congested highway leads to negative externalities: for other commuters, such as longer waiting hours, mental distress, and more fuel costs, and the rest of the society, as air pollution. Therefore, a toll on a congested road is a corrective tax, preventing overuse beyond the efficient level. As might be expected, in Istanbul, too, vehicles are tolled on the strait since the first bridge's inception.

Our research study is spurred by the changes brought in by the opening of the Eurasian tunnel. Before then, the commuter demand to cross the strait was served by the two bridges, which the government owns and operates via its agency, Turkey's General Directorate of Highways (GDH). The Eurasian tunnel, however, was funded by private enterprise under the build-operate-transfer model, and it now has a private operator. This does not mean that its private operator has complete control over the tunnel's toll scheme. It has "guaranteed minimum revenue" and "profit-sharing for parts exceeding guaranteed revenue" arrangements with the government. Therefore, the tunnel's terms of operation are settled in negotiations between the government and the private operator.

[1] *In 2016, a third bridge across the strait has become operational, too. But in the rest of this paper, we focus our attention on the two bridges and the Eurasian tunnel. We assume away the third bridge for two reasons: Frist, the third bridge is located in the sparsely populated north of the city, and it is primarily used for transit traffic by buses and long vehicles rather than by Istanbulites. Second, assuming the third bridge helps simplify our analysis without losing the main insights.*

For many years, the government has used a unidirectional toll regime on the two bridges, meaning that vehicles are tolled only when they cross Europe to Asia. Indeed, tolling vehicles only in one direction made much sense until 2016: When alternative crossings were absent, a car crossing the two bridges in one direction had to come back also by using the two bridges. Therefore, tolling a vehicle T Liras in each direction was equivalent in revenue to tolling the vehicle $2T$ Liras only from Europe to Asia. But the unidirectional toll regime had its advantages: It helped eliminate the need to install and operate tolling equipment from Asia to Europe. In this direction, the traffic flow was not impeded by tolling.

But the opening of the Eurasian tunnel now casts doubt on the effectiveness of a unidirectional toll regime on the two bridges. Since its inception, in the tunnel, vehicles are tolled at a fixed rate in both directions. Furthermore, the tunnel's toll rate is much higher than the toll rate on the two bridges. To put our discussion in perspective, Table 1 below presents the toll rates in 2021 for standard automobiles in the tunnel and on the two bridges.

Table: 1
Toll Rates for A Standard Automobile on the Two Bridges and in the Tunnel

	Europe → Asia	Asia → Europe	Roundtrip
Two bridges	13.25 Liras	toll-free	13.25 Liras
The Eurasian tunnel	46.00 Liras	46.00 Liras	92.00 Liras
Premium paid for the Eurasian tunnel	32.75 Liras	46.00 Liras	

As seen in Table 1, commuters face asymmetric incentives under the effective toll schemes when they cross from Europe to Asia and from Asia to Europe: The premium paid for the tunnel is 32.75 Liras from Europe to Asia and 46.00 Liras from Asia to Europe. Public criticism is also that the tunnel's toll rate is too high. The opening of the Eurasian tunnel raises several public policy questions: One question is about the fair distribution of toll revenue. Note that under the effective toll schemes, a commuter who crosses from Europe to Asia via the tunnel may divert to the two bridges in the opposite direction since from Asia to Europe, the bridges are toll-free. But this commuter does not pay a toll to the government, although in her roundtrip, she uses both the tunnel and one of the two bridges. But the more important question is about the efficiency of the distribution of the strait's traffic load. The premium paid for the tunnel is too high, even in the direction from Europe to Asia. Arguably, this situation causes too many vehicles to divert to the bridges, leading to overcongestion (congestion beyond the efficient level) and loss in social welfare.

This paper aims to study the above issues and offer guidance on public policy. To this end, we introduce a transportation model. Our model is simple, yet it captures the problem's main ingredients.

We assume that there are two crossings connecting the two sides of a city, A and E- as per Asia and Europe. Crossing 1 is privately operated, as per the Eurasian tunnel, and crossing 2 is publicly operated, as per the two bridges[2]. The two crossings may have different capacities. For instance, in Istanbul, in one direction, there are seven lanes on the two bridges and two lanes in the tunnel, meaning that the capacity to carry the traffic load is higher on the two bridges.

We assume a continuum of commuters who wish to transport from side A to E. We also assume that the model is symmetrically applicable in the opposite direction. Our model captures the heterogeneity in commuter preferences by assuming that commuters are of two types: type 1 commuters, whose favourite crossing is 1, and type 2 commuters, whose favourite crossing is 2. We assume that crossing 1 is “overpreferred” because if each type uses its favourite crossing, its traffic density (the mass of commuters divided by capacity) becomes higher. In the Bosporus, arguably, the Eurasian tunnel is the preferred crossing because its capacity is smaller, and it is located in the densely populated southern part of the city.

In our model, the incentives faced by a commuter are captured through three cost items: the rates of toll, the cost of diversion, and the cost of congestion. A commuter using crossing i pays its rate of toll T_i. If i is not her favourite crossing, she also incurs a diversion cost since she is diverting from her optimal route. Finally, the commuter incurs a congestion cost. We assume that the rate of congestion cost is a convex function of traffic density. This assumption is in line with the triangular traffic flow-density curve, commonly used in the transportation literature. In our model, each commuter uses the crossing that minimises her aggregate cost. For simplicity, we assume that the demand to cross the strait is inelastic (i.e., every commuter crosses the strait).

The focus of our analysis is (social) welfare maximisation. Since tolls are transferred payments, welfare is maximised when the sum of the diversion and congestion costs is minimised. Let μ^w denote the welfare-maximizing allocation. Let T_1 and T_2 be the rates of tolls for crossings 1 and 2. Thus, $T_1 - T_2$ is the premium paid for using the overpreferred crossing. Among our findings, the two results that are of greatest practical value are as follows: In Proposition 1, we show that under μ^w, the traffic density is higher at the overpreferred crossing (crossing 1). In Proposition 2, we show that there is an optimum premium level that induces the allocation μ^w.

Simple as they may seem, Propositions 1 and 2 have important implications regarding the design of the Bosporus toll schemes. They show that for welfare maximisation, the premium paid for the Eurasian tunnel should be the same in the two directions and not be set at an excessive rate. The current practice, however, is diametrically opposed to this finding. Under the existing toll schemes, the tunnel’s premium rate is very high and is not

[2] *We do not introduce a separate crossing for each bridge in our model since doing so will complicate our analysis without changing the main insights.*

the same in the two directions. In light of our findings, we recommend lowering the premium paid for the tunnel and setting the same in the two directions[3].

In current practice, as explained above, the government tolls vehicles only on the bridges when they cross Europe to Asia. It is worth mentioning that the government need not abandon this historical practice. The strait’s traffic flow can optimally be distributed across the tunnel and the two bridges while continuing with the historical practice of tolling vehicles only when they cross Europe to Asia. As an illustration of this point, for the tunnel, suppose that the optimum premium rate is 20 Liras. Also, as in Table 1, suppose that the government wants to toll vehicles 13.25 Liras for a roundtrip on the bridges. Then, the optimal traffic distribution can be achieved in both directions under the following toll regime: On the bridges, vehicles are tolled 13.25 Liras from Europe to Asia and not tolled from Asia to Europe. In the tunnel, vehicles are tolled 33.25 Liras from Europe to Asia (=13.25+20) and 20 Liras from Asia to Europe (=0+20).

The rest of the paper is organised as follows: Section 2 presents the relevant literature. Section 3 introduces our model. Section 4 presents our results: In Section 4.1, we identify the welfare-maximizing allocation. Section 4.2 studies the implementation of the welfare-maximizing allocation as an equilibrium outcome. In Section 4.3, we obtain closed-form solutions in our model under the simplifying assumption that the rate of the congestion cost function is linear. In Section 5, we conclude with a summary of our findings. In Appendix A, we present the triangular traffic flow-density curve and justify our assumption that the rate of congestion function is convex. In Appendix B, we present two proofs omitted from the main text.

2. Related Literature

In the analysis of our model, we consider two equilibrium notions: In a “regulated equilibrium,” we assume that the government controls the toll rates for both crossings. In an “unregulated equilibrium,” crossing 1’s toll rate is set by its private operator under the profit-maximization motive. Our analysis based on the unregulated equilibrium notion relates our study to the literature on mixed-oligopoly markets. In a mixed-oligopoly market, several profit-maximising firms compete against a welfare-maximizing public enterprise, as assumed in our model under the unregulated equilibrium notion.

Mixed-oligopoly markets are prevalent worldwide in various sectors such as healthcare, insurance, banking, energy, steel, postal service, and telecommunications. The studies in this literature considered the impact on the welfare of the presence of the public enterprise (Anderson et al., 1997; de Fraja & Delbono, 1989; Ishibashi & Matsumura, 2006),

[3] *The first draft of this article was submitted in late 2021. After the submission, in January 2022, the tolling regime for Bosporus crossings in Istanbul was amended. In addition to the usual annual inflation adjustments in toll rates, just as for the Eurasian tunnel, on the bridges too, vehicles began to be tolled in both directions at a fixed rate. Consequently, the premium paid for using the tunnel was fixed in the two directions, which is in line with one of the main conclusions of our analysis in this paper.*

of the partial privatisation of the public enterprise (Fujiwara, 2007; Matsumura & Kanda, 2005), of the absence of the entry barriers (Anderson et al., 1997; Matsumura & Kanda, 2005), and collective bargaining by public employees (Ishibashi & Matsumura, 2006). For a survey on mixed-oligopoly games, see Bös (2015). For older surveys, see Condon (1994), Nett (1993), and De Fraja and Delbono (1989)[4].

The transportation model in our paper is customised and tailored to study optimal toll design for Istanbul's Bosporus crossings. Therefore, in specific dimensions, it differs from earlier mixed-oligopoly studies: In our setting, firms (operators of crossings) have zero-cost functions, the number of firms is fixed, and the goods are heterogeneous (i.e., commuters have varying preferences over the two crossings). Furthermore, in our setting, firms engage in a two-way pricing scheme (i.e., they toll vehicles in both directions). However, the most characteristic feature in our setting is that the level of demand influences a consumer's payoff: a commuter's derived utility from using a crossing decrease when its congestion level (i.e., its demand level) is higher.

We should also note that there is a line of research in the game theory literature that studies the congestion of resources. Like our model, self-interested agents route traffic through a congested network in routing games. The congestion level on the network's edge increases with the number of agents travelling through that edge. For studies on routing games, see Chapter 18 in Nisan et al. (2007) and the references therein. Yet, in these studies, resources are free for the users, as opposed to in our setting where commuters pay to use the resources (i.e., the two crossings).

3. Model

Let A and E be the two sides of a city, as per Asia and Europe in Istanbul. We assume that there is a continuum of commuters $[0,1)$, each with an infinitesimal mass, who wish to travel from side A to E. Our model is equally applicable when commuters are to travel in the opposite direction.

Let 1 and 2 be the two crossings that connect the two sides. Crossing 1 is intended to stand for the Eurasian tunnel in Istanbul. Crossing 2 is intended to stand for the two bridges.

For commuters, crossings 1 and 2 are imperfect substitutes: Under ceteris paribus conditions (i.e., under similar toll rates and congestion levels), some commuters prefer crossing 1 over 2, and others prefer crossing 2 over 1. For instance, a commuter will find it more convenient to use crossing 1 if her home and workplace on sides A and E are closely

[4] *For other mixed-oligopoly studies, see Merrill and Schneider (1966), Harris and Wiens (1980), Estrin and De Meza (1995), Cremer et al. (1989), Matsushima and Matsumura (2003), and Casadesus-Masanell and Ghemawat (2006). For mixed-oligopoly studies in the transportation literature, see Qin et al. (2017), Czerny et al. (2014), Mantin (2012) and Yang and Zhang (2012).*

located to the endpoints of crossing 1. A route via crossing 2 then takes longer, leading to increased travel time and fuel expenses.

The heterogeneity in commuter preferences is embodied in our model: We call the commuters in the intervals $[0, x)$ and $[x, 1)$, in order, *type 1* and *type 2* commuters. A type i commuter's favourite crossing is crossing i, and she incurs a *diversion cost* if forced to "divert" to her non-favourite crossing. We assume that the rate of diversion cost, $c > 0$, is constant. Thus, a commuter who diverts to her non-favourite crossing incurs a diversion cost equal to her "infinitesimal mass" multiplied by c. This commuter incurs no diversion cost if she uses her favourite crossing.

A crossing's *capacity* helps determine the traffic load it can carry without causing much traffic congestion. One way to interpret the capacity of a crossing is as being proportional to its number of lanes. Nevertheless, other factors may also play a role in determining a crossing's capacity, such as the road quality and the capacities of the road networks connected to the crossing's entry and exit points. We normalise the sum of capacities of the two crossings to be 1. Let $k \in (0,1)$ and $1 - k$ be, in order, the capacities of crossings 1 and 2.

The *traffic density* at a crossing is the ratio of the total mass of commuters using that crossing to its capacity.

We assume that crossing 1 is *overpreferred*, and crossing 2 is *underpreferred*, in the sense that $x > k$. In other words, we assume that if every commuter uses her favourite crossing, crossing 1's traffic density, $\frac{x}{k}$, will be greater than crossing 2's traffic density, $\frac{1-x}{1-k}$.

Notice that the assumption that crossing 1 is overpreferred is in line with the presumption that under ceteris paribus conditions (i.e., if toll rates were similar), the Eurasian tunnel would be in high demand by Istanbulites. But the analysis in our paper is applicable even if this supposition is wrong: If the Eurasian tunnel is the underpreferred crossing, we can interpret crossing 2 as the Eurasian tunnel and crossing 1 as standing for the two bridges.

Congestion means the traffic density is too high. In congested traffic, travel times and fuel expenses are higher, and commuters suffer mental distress. We group such expenses incurred due to congested traffic under the heading *congestion cost*. The congestion cost at a crossing depends on the crossing's traffic density. We assume that the rate of congestion cost is a function φ of the traffic density t. Thus, a commuter incurs a congestion cost equal to her "infinitesimal mass" multiplied by $\varphi(t)$ when the traffic density is t.

Our analysis builds on certain assumptions about model parameters and commuter behaviour. For expositional ease, we will introduce these assumptions as they are needed. Our first assumption is about the rate of congestion function φ. As given below, we assume that φ is convex.

Assumption 1: (**convexity**) $\varphi'(t) > 0$ and $\varphi''(t) \geq 0$ for all t.

Assumption 1 can be justified by the "triangular view" commonly held in the transportation literature. The triangular view pertains to the traffic flow, defined as the number of vehicles per minute crossing a reference point on the road. The traffic flow is a function of the traffic density. According to the triangular view, the traffic flow increases at a linear rate up to some critical level of traffic density, t^*, and then it decreases at a linear rate above the critical level t^*. When the traffic density is below t^*, it is called the "free-flow phase," and when it is above t^*, it is called the "congestion phase." For the interested reader, in Appendix A, we show that in the congestion phase, the time spent on a journey increases convex as traffic density increases. This justifies Assumption 1 in the congestion phase. In the free-flow phase, however, Assumption 1 would not hold, and the results in our paper are not applicable. Nevertheless, Istanbul's Bosporus traffic is most congested, so our analysis is suitable for most of the day (except for the overnight traffic from, say, from 2:00 am until 6:00 am)[5].

Our second assumption pertains to commuter behaviour. As given below, we carry out our analysis under a "covered market" assumption, which states that commuters always use one or the other crossing.

Assumption 2: (**covered market**) Each commuter uses the crossing for which the aggregate cost that she incurs is smaller. In the case of equality, she uses crossing 2[6].

Put differently; the covered market assumption assumes that the commuter demand to cross the strait is *inelastic*. It presumes that for each commuter, the willingness to travel is sufficiently high, or the aggregate cost rate is low enough so that she never opts out of a journey. In the Bosporus setting, Assumption 2 can be justified on two grounds. First, for the most part, Istanbulites cross the strait regularly for business purposes rather than for pleasure. This type of journey can be seen as a "necessity," not very sensitive to the costs of toll and congestion. Second, for various reasons, the government does not set a prohibitively high toll rate on the two bridges-one that would lead to a meaningful fall in demand. Therefore, our covered market assumption is arguably a good approximation of the real-life situation in Bosporus traffic. We should also note that Assumption 2 helps simplify our analysis and obtain closed-form solutions from a modelling perspective.

The two crossings have separate operators. Let operator i be the operator of crossing i. We assume that crossing 1's operator is private (as per the Eurasian tunnel) and crossing 2's operator is public (as per the two bridges).

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5 *For studies on the triangular view, see Saberi and Mahmassani (2012), Geroliminis and Sun (2011), Cassidy et al. (2011), and Geroliminis and Daganzo (2008).*

6 *In Assumption 2, the presumption that a commuter uses crossing 2 when the aggregate cost is the same is an innocuous one, one that brings expositional ease in our analysis.*

Let T_i be the rate of the toll that operator i charges a commuter using crossing i. Thus, if a commuter uses crossing i, she incurs a toll equal to her “infinitesimal mass” multiplied by T_i. The amount $T_1 - T_2$ is the *premium* that the commuter pays for using the overpreferred crossing (crossing 1).

The rate of the aggregate cost incurred by a commuter is equal to the sum of the rates of diversion cost, congestion cost, and the toll that she incurs. For instance, if d and $1-d$ are, respectively, the total masses of commuters using crossings 1 and 2, then the rate of the aggregate cost incurred by a commuter of type j using crossing i is:

- $T_1 + \varphi\left(\frac{d}{k}\right)$, for $j = 1$ and $i = 1$
- $T_1 + c + \varphi\left(\frac{d}{k}\right)$, for $j = 2$ and $i = 1$
- $T_2 + c + \varphi\left(\frac{1-d}{1-k}\right)$, for $j = 1$ and $i = 2$
- $T_2 + \varphi\left(\frac{1-d}{1-k}\right)$, for $j = 2$ and $i = 2$

The aggregate cost incurred by a commuter is equal to her “infinitesimal mass” multiplied by the rate of aggregate cost that she incurs.

An allocation specifies the crossing used by each commuter. Under optimal commuter behaviour (Assumption 2), all type 1 commuters use crossing 1, or all type 2 commuters use crossing 2. Therefore, we can define an allocation simply as follows: An *allocation* is a number $\mu \in [0,1]$, with the interpretation that under μ, the commuters in the interval $[0, \mu)$ use crossing 1, and the commuters in the interval $[\mu, 1)$ use crossing 2. Notice that if $\mu = 0$, every commuter uses crossing 2, and for $\mu = 1$, every commuter uses crossing 1.

Let $\mu(T_1, T_2)$ be the allocation induced under optimal commuter behavior when the rates of tolls are T_1 and T_2[7]. Then, operator 1’s toll revenue is $T_1\mu(T_1, T_2)$, and operator 2’s toll revenue is $T_2\big(1 - \mu(T_1, T_2)\big)$. We assume that operating costs are negligible, and hence, an operator’s profit is equal to its toll revenue. But the results in our paper remain unchanged if operating costs were non-negligible but fixed. Because in that case, too, operators would be maximising their profits by maximising their revenues.

4. Results

4.1. Welfare Maximisation

In this subsection, we identify the allocation that maximises social welfare.

[7] *The allocation $\mu(T_1, T_2)$ is unique. We leave the easy proof to the interested reader.*

Welfare maximisation necessitates minimising the costs to society. Note that tolls are transferred payments from commuters to the operators of the two crossings. Therefore, the social costs include only the total congestion and diversion costs.

Let $TDC(\mu)$, $TCC(\mu)$, and $SC(\mu)$ denote, respectively, the total diversion cost, the total congestion cost, and the social cost under the allocation μ. Then:

$$TDC(\mu) = c\,|\mu - x|$$

$$TCC(\mu) = \mu\,\varphi\left(\frac{\mu}{k}\right) + (1-\mu)\varphi\left(\frac{1-\mu}{1-k}\right)$$

$$SC(\mu) = TDC(\mu) + TCC(\mu)$$

With some algebra, one can show that the first two derivatives of the function $TCC(\mu)$ are as follows:

$$TCC'(\mu) = \left[\varphi\left(\frac{\mu}{k}\right) - \varphi\left(\frac{1-\mu}{1-k}\right)\right] + \left[\frac{\mu}{k}\varphi'\left(\frac{\mu}{k}\right) - \frac{1-\mu}{1-k}\varphi'\left(\frac{1-\mu}{1-k}\right)\right]$$

$$TCC''(\mu) = \left[\frac{2}{k}\varphi'\left(\frac{\mu}{k}\right) + \frac{2}{1-k}\varphi'\left(\frac{1-\mu}{1-k}\right)\right] + \left[\frac{\mu}{k^2}\varphi''\left(\frac{\mu}{k}\right) + \frac{1-\mu}{(1-k)^2}\varphi''\left(\frac{1-\mu}{1-k}\right)\right]$$

By Assumption 1, we have $\varphi' > 0$ and $\varphi'' \geq 0$. Therefore, we obtain that $TCC'' > 0$. Furthermore, when we plug in $\mu = k$, we get $TCC'(k) = 0$. We present these findings in Lemma 1.

Lemma 1: Under Assumptions 1 and 2, $TCC'' > 0$ and $TCC'(k) = 0$.

Let μ^w denote the welfare-maximizing allocation: i.e., $SC(\mu)$ is minimized for $\mu = \mu^w$. Also, let μ^d and μ^c denote, in order, the allocations that minimise the total diversion cost and the total congestion cost. Proposition 1 characterises these allocations.

Proposition 1: Under Assumptions 1 and 2, we have:

$\mu^d = x, \mu^c = k$

$\mu^w = x$ if $TCC'(x) \leq c$

$\mu^w \in (k, x)$ and μ solves $TCC'(\mu^w) = c$ if $TCC'(x) > c$

Proof

The total diversion cost is zero and minimised when each commuter uses her favourite crossing. Thus, $\mu^d = x$. Also, by Lemma 1, we get $\mu^c = k$.

Let $\mu > x$. Then, $TDC'(\mu) = c > 0$. Also, since $\mu > k$, by Lemma 1, $TCC'(\mu) > 0$. Then, $SC'(\mu) = TDC'(\mu) + TCC'(\mu) > 0$. But then the allocation μ cannot be welfare-maximizing. Thus, $\mu^w \notin (x, 1]$.

Suppose that $TTC'(x) \leq c$. Let $\mu < x$. Then, $TDC'(\mu) = -c$. Note that, by Lemma 1, $TCC'' > 0$. Since $TTC'(x) \leq c$ and $\mu < x$, the fact that $TCC'' > 0$ implies that $TTC'(\mu) < c$. Then, $SC'(\mu) = TDC'(\mu) + TCC'(\mu) < 0$. But then the allocation μ cannot be welfare-maximizing. Thus, we get $\mu^w = x$.

Suppose that $TCC'(x) > c$. By Lemma 1, we have $TCC'(k) = 0$ and $TCC'' > 0$. These facts imply that there exists an allocation $\mu^* \in (k, x)$ such that $TCC'(\mu^*) = c$. Since $\mu^* < x$, we also get $TDC'(\mu^*) = -\text{c}$. Thus, we get $SC'^{(\mu^*)} = TDC'(\mu^*) + TCC'(\mu^*) = 0$. Thus, $SC(\mu)$ is minimized for $\mu = \mu^*$. Thus, we have $\mu^w = \mu^*$. This completes our proof.

Proposition 1 states that the total diversion cost is minimises when each type uses its favourite crossing. It also says that the total congestion cost is minimised when the traffic distribution is balanced (i.e., when the two crossings have the same traffic density).

We say that the traffic distribution is *balanced* if the two crossings have the same traffic density (i.e., when they are equally congested). Note that the traffic density is balanced for $\mu = k$. As given in Proposition 1, the total congestion cost is minimised when the traffic distribution is balanced (i.e., $\mu^c = k$).

Finally, note that Proposition 1 states that $\mu^w \in (k, x]$, meaning that the welfare-maximizing allocation leads to unbalanced traffic distribution. Proposition 1 says that the traffic density should be higher at the overpreferred crossing (i.e., crossing 1) for welfare maximisation. Suppose the Eurasian tunnel is the overpreferred crossing. In that case, the policy implication is as follows: For social welfare maximisation, the traffic density in the Eurasian tunnel should not be less than that on the two bridges. Therefore, it is not optimal to set too high a toll rate on the Eurasian tunnel that would divert away too many commuters to the two bridges.

For $\mu = x$, each commuter type uses its favourite crossing. Therefore, we call $\mu = x$ a *separating allocation*. In our model, welfare can indeed be maximised under a separating allocation (i.e., $\mu^w = x$). This situation occurs if the rate of diversion cost is so excessive that the diversion cost needs to be eliminated for welfare maximisation. But this situation is not in line with the spirit of our analysis since we aim to capture the tradeoff that the social planner faces in balancing out the costs of diversion and congestion. Therefore, in the rest of the paper, we will proceed under Assumption 3, which guarantees that welfare is not maximised under a separating allocation.

Assumption 3: (**c is not excessive**) $c < TCC'(x)$.

4.2. Implementation of the Welfare-Maximizing Allocation

This section studies how the welfare-maximizing allocation μ^w can be implemented. In other words, we study how the social planner can induce μ^w as an equilibrium outcome. We consider two notions of equilibrium notions, which we introduce next.

Let $br(T_2) = \arg\max T_1\mu(T_1, T_2)$ be operator 1's best response function (possibly multi-valued). That is, given T_2, operator 1 maximizes its profit by setting $T_1 \in br(T_2)$. The two equilibrium notions that we consider are as follows.

Definition: A triplet $\langle T_1, T_2, \mu(T_1, T_2)\rangle$ is called a "regulated equilibrium." A triplet $\langle T_1, T_2, \mu(T_1, T_2)\rangle$ is called an "unregulated equilibrium" if $T_1 \in br(T_2)$.

A few words are in place to give insight into our above definitions. Crossing 2 is publicly operated, so we presume that the social planner sets its rate. But the social planner may or may not have the power to regulate the toll rate for crossing 1, which is privately operated. Above, our regulated equilibrium notion presumes this prerogative for the social planner, and our unregulated equilibrium notion does not. Under our unregulated equilibrium notion, we assume that crossing 1's toll rate is set by its private operator under the profit-maximization motive.

The above two equilibrium notions correspond to two different perspectives regarding the operation of the Bosporus crossings in Istanbul. Since the bridges are under public operation (as per crossing 2 in our model), the government directly sets their toll rate. The Eurasian tunnel, however, is privately operated (as per crossing 1 in our model). The tunnel's terms of operation are negotiated between the government and its private operator. While the government has some influence regarding the tunnel's toll scheme, the extent of this influence is contestable. Therefore, in our analysis, we consider both scenarios.

The following two propositions identify what toll schemes induce the welfare-maximizing allocation as the outcome of a regulated and an unregulated equilibrium. The proofs are easy and left to the reader.

Proposition 2: Under Assumptions 1-3, $\langle T_1, T_2, \mu(T_1, T_2)\rangle$ is a regulated equilibrium such that $\mu(T_1, T_2) = \mu^w$ if:

$$T_1 - T_2 = c - \left[\varphi\left(\frac{\mu^w}{k}\right) - \varphi\left(\frac{1-\mu^w}{1-k}\right)\right]$$

Since $\mu^w \in (k, x)$, we also get $T_1 - T_2 < c$.

Proposition 3: Under Assumptions 1-3, $\langle T_1, T_2, \mu(T_1, T_2)\rangle$ is an unregulated equilibrium such that $\mu(T_1, T_2) = \mu^w$ if:

$$T_1 - T_2 = c - \left[\varphi\left(\frac{\mu^w}{k}\right) - \varphi\left(\frac{1-\mu^w}{1-k}\right)\right] \text{ and } T_1 \in br(T_2).$$

Since $\mu^w \in (k, x)$, we also get $T_1 - T_2 < c$.

Simple as they may be, Propositions 2 and 3 have important implications for our Bosporus setting. To emphasise these implications, we present below Corollaries 1 and 2. But before that, we need to introduce some new terminology and present Assumption 4.

One question pursued in our analysis is how the toll schemes should be set in two directions in our Bosporus setting: i.e., when crossing from Asia to Europe and from Europe to Asia. In current practice, in the Eurasian tunnel, vehicles are tolled in both directions and at the same rate, and on the two bridges, only when they cross from Europe to Asia. For two-directional toll schemes, we introduce below the term "toll regime" and its special cases.

Definition: A "toll regime" is a four-tuple $\langle T_1, T_2, T_1', T_2' \rangle$, with the interpretation that the rates of tolls for crossings 1 and 2 are, in order, T_1 and T_2 from side A to E, and $T_1^{'}$ and $T_2^{'}$ from side E to A. We call $\langle T_1, T_2, T_1', T_2' \rangle$ a "unidirectional toll regime" if $T_2 = 0$. We call $\langle T_1, T_2, T_1', T_2' \rangle$ a "simple unidirectional toll regime" if $T_2 = 0$ and $T_1 = T_1'$.

Note that under a "unidirectional toll regime," at crossing 2, vehicles are tolled only in one direction (from side E to A). Under a "simple unidirectional toll regime," additionally, crossing 1's toll rate is set the same in both directions. In current practice, a simple unidirectional toll regime is used for Bosporus crossings.

We are concerned with the "implementation" of the welfare-maximizing allocation. In other words, we inquire when the welfare-maximizing allocation is obtained as an equilibrium outcome. Below, we introduce these notions formally.

Definition: A toll regime $\langle T_1, T_2, T_1', T_2' \rangle$:

- implements the welfare-maximizing allocation in regulated equilibria if $\langle T_1, T_2, \mu^w \rangle$ and $\langle {T_1}', T_2', \mu^w \rangle$ are regulated equilibria
- implements the welfare-maximizing allocation in unregulated equilibria if $\langle T_1, T_2, \mu^w \rangle$ and $\langle {T_1}', T_2', \mu^w \rangle$ are unregulated equilibria

Note that we introduced our transportation model assuming that commuters wish to travel from side A to E. Our following assumption thinks that our transportation model is symmetrically applicable in the opposite direction-i.e., when commuters travel from side E to A.

Assumption 4: (**symmetry**) The same transportation model is applicable in the two directions: i.e., when commuters are to travel from side A to E and from side E to A.

Assumption 4 is not unrealistic when considered in our Bosporus setting. In Istanbul, absent alternative means, each commuter who crosses from Asia to Europe (or from Europe to Asia) eventually crosses back in the reverse direction. And most often, in their reverse journeys, commuters use the exact opposite route, such as when they travel between home and workplace on the two sides. Therefore, our symmetry assumption arguably holds in our Bosporus setting. But we should caution that while the commuters' transportation needs in the two directions may be the same in the aggregate, they may not be the same at a given point in time. For instance, in Istanbul, there are more workplaces on the European side. Consequently, the traffic flow is heavier from Asia to Europe in the morning and from

Europe to Asia in the evening. Arguably, a dynamic toll regime can more effectively distribute the traffic flow over time, under which toll rates increase with traffic density. But the authorities prefer to use a static toll regime in Istanbul, where toll rates are the same throughout the day. Therefore, in this paper, we abstract from dynamic considerations. And when attention is restricted to static toll regimes, our assumption that the city's transportation needs are symmetric in the two directions is realistic.

Corollaries 1 and 2 below follow Propositions 2 and 3 in order. They present the limitations regarding welfare maximisation if attention is restricted to unidirectional or simple unidirectional toll regimes.

Corollary 1: Under Assumption 1-4, there exists no simple unidirectional toll regime that implements μ^w in regulated equilibria. A unidirectional toll regime $\langle T_1, T_2 = 0, T_1', T_2' \rangle$ implements μ^w in regulated equilibria if

$$T_1 = T_1' - T_2' = c - \left[\varphi\left(\frac{\mu^w}{k}\right) - \varphi\left(\frac{1-\mu^w}{1-k}\right)\right].$$

Corollary 2: Under Assumption 1-4, there exists a unidirectional toll regime that implements μ^w in unregulated equilibria only if

$$c - \left[\varphi\left(\frac{\mu^w}{k}\right) - \varphi\left(\frac{1-\mu^w}{1-k}\right)\right] \in br(0).$$

The current toll regime for Bosporus crossings is a simple unidirectional one. Corollary 1 states that this toll regime cannot be welfare-maximizing. Under the current toll regime, commuters pay different premiums for the Eurasian tunnel in the two directions. But Corollary 1 states that there is a unique optimal premium, meaning that the current toll regime does not implement the welfare-maximizing allocation at least in one direction.

According to Corollary 1, the authorities must give up using a simple unidirectional toll regime for welfare maximisation. But note that they do not need to give up using a unidirectional toll regime. According to Corollary 1, on the bridges, vehicles can be tolled only from Europe to Asia, as in current practice, as long as the premium paid for the Eurasian tunnel is set the same and equal to its optimal level in both directions. For instance, if the optimal level of the premium is ₺20, and if the authorities want a toll rate of ₺10 for a roundtrip on the two bridges, this can be achieved under the following unidirectional toll regime as follows:

$$T_1 = 20, T_2 = 0, T_1' = 30, T_2' = 10$$

Corollary 1, studying regulated equilibria, identifies the optimal toll regime under the assumption that the authorities control all toll rates. However, if the Eurasian tunnel's toll rate is set by its private operator under the profit-maximization motive, we need to turn our attention to unregulated equilibria. Unfortunately, in an unregulated equilibrium, welfare is unlikely to be maximised under a unidirectional toll regime. Given the bridge's toll rate, its private operator sets the Eurasian tunnel's toll rate at its profit-maximising level in an unregulated equilibrium. This premium level need not be welfare-maximizing. The government can finetune the toll rate for the bridges so that the induced premium level becomes optimal. But this "finetuned" toll rate is unlikely to be zero, as emphasised in Corollary 2. Therefore, if the tunnel's operator acts under the profit-maximization motive for welfare maximisation, the authorities may have to abandon using a unidirectional toll scheme and begin to toll vehicles in both directions.

4.3. Reduced Model

In this section, to get closed-form solutions, we simplify the assumption that the rate of congestion cost function φ is linear.

Assumption 5: (φ is linear)$\varphi(t) = \pi t, \pi > 0$.

Under Assumption 5, note that:

$$TDC(\mu) = c\ |\mu - x|$$

$$TCC(\mu) = \frac{\pi}{k}\mu^2 + \frac{\pi}{1-k}(1-\mu)^2$$

$$SC(\mu) = TDC(\mu) + TCC(\mu)$$

First, we solve for the allocation that maximizes welfare.

Proposition 4: Under Assumptions 1-3 and 5, we have

$$\mu^w = k + \frac{c}{2k}k(1-k)$$

Therefore, under μ^w, the traffic densities at crossings 1 and 2 are, respectively, $1 + \frac{c}{2k}(1-k)$ and $1 - \frac{c}{2\pi}k$.

Proof

By Proposition 1, we should solve for $TCC'(\mu) = c$ to find $\mu = \mu^w$. Then:

$$2\frac{\pi}{k}\mu^w - 2\frac{\pi}{1-k}(1-\mu^w) = c$$

$$\Rightarrow \mu^w = k + \frac{c}{2\pi}k(1-k)$$

Note that the welfare-maximizing allocation induces an unbalanced traffic distribution. According to Proposition 4, under μ^w, the overpreferred crossing's traffic density is higher, and the unbalance grows as the ratio c/k increases. This result is expected: Under $\mu^c = k,$ the total congestion cost is minimised. Under $\mu^d = x$, the total diversion cost is minimized. As the ratio c/k increases, the diversion cost's weight in the social cost increases, bringing μ^w closer to μ^d. Therefore, as the ratio c/k increases, the overpreferred crossing’s traffic density increases under the welfare-maximizing allocation.

In our next result, we identify the regulated equilibria.

Proposition 5: Under Assumptions 1-3 and 5, in a regulated equilibrium $\langle T_1, T_2, \mu(T_1, T_2)\rangle$, we have

$$\mu(T_1,T_2) = \begin{cases} 1 & \text{for } T_1 \leq B_1 \\ k-(T_1-T_2+c)\frac{k(1-k)}{\pi} & \text{for } B_1 < T_1 < B_2 \\ x & \text{for } B_2 \leq T_1 \leq B_3 \\ k-(T_1-T_2-c)\frac{k(1-k)}{\pi} & \text{for } B_3 < T_1 < B_4 \\ 0 & \text{for } B_4 \leq T_1 \end{cases}$$

where:

$$B_1 = T_2 - c - \pi\frac{1}{k}, B_2 = T_2 - c - \pi\ \frac{x-k}{k(1-k)}$$

$$B_3 = T_2 + c - \pi\ \frac{x-k}{k(1-k)},\ B_4 = T_2 + c + \pi\frac{1}{1-k}$$

Proof

See Appendix B.

Note that for fixed T_2, the function $\mu(T_1, T_2)$, given in Proposition 5, is continuous in T_1. (The reader may verify this by checking the value of the function at corner points.)

Our following two results show how the social planner can induce the welfare-maximizing allocation as the outcome of a regulated and an unregulated equilibrium.

Proposition 6: Under Assumptions 1-3 and 5, $\langle T_1, T_2, \mu(T_1, T_2)\rangle$ is a regulated equilibrium such that $\mu(T_1, T_2) = \mu^w$ if the premium $T_1 - T_2$ is equal to $\frac{c}{2}$.

Proof

Using Proposition 2 and Assumption 5, we get:

$$T_1 - T_2 = c - \left[\pi\frac{\mu^w}{k} - \pi\frac{1-\mu^w}{1-k}\right]$$

$$\Rightarrow \mu^w = k - \frac{1}{\pi}(T_1 - T_2 - c)k(1-k)$$

When we substitute for μ^w using Proposition 4, we get $T_1 - T_2 = \frac{c}{2}$.

Proposition 7: Under Assumptions 1-3 and 5, $\langle T_1, T_2, \mu(T_1, T_2)\rangle$ is an unregulated equilibrium such that $\mu(T_1, T_2) = \mu^w$ if:

$T_2 = \frac{\pi}{1-k}$ and $T_1 = \frac{c}{2} + T_2$

Proof

See Appendix B.

Corollaries 3 and 4 below follow Propositions 6 and 7. Under Assumption 5, they present the limitations regarding welfare maximisation if attention is restricted to unidirectional or simple unidirectional toll regimes.

Corollary 3: Under Assumptions 1-5, a unidirectional toll regime implements μ^w in regulated equilibria if $T_1 - T_2 = T_1' - T_2' = \frac{c}{2}$.

Corollary 4: Under Assumptions 1-5, the following toll regime implements μ^w in unregulated equilibria: $\langle T_1, T_2, T_1', T_2'\rangle$ such that $T_2 = T_2' = \frac{\pi}{1-k}$ and $T_1 = T_1' = \frac{c}{2} + T_2$.

Above, thanks to our simplifying assumption, we obtained closed-form solutions for optimal toll regimes. But otherwise, our findings are similar to the one in the preceding subsection: Corollary 3 states that a multiplicity of unidirectional toll regimes implement μ^w in regulated equilibria. But note that no simple unidirectional toll regime implements μ^w in a regulated equilibrium. Corollary 4 states that this conclusion is foregone if crossing 1’s toll scheme is set by its private operator under the profit-maximization motive. Because there is a unique toll regime that implements μ^w in an unregulated equilibrium, which is not a unidirectional toll regime.

5. Summary

Divided by a narrow strait called the Bosporus, Turkey’s megacity Istanbul is located half in Asia and half in Europe. Commuter demand to cross the strait is naturally high, which has recently been served by two toll bridges. To relieve the traffic on the strait, an underground connection has been recently launched, the so-called Eurasian tunnel, which has a private operator. In current practice, on the bridges, vehicles are tolled at a low rate and only from Europe to Asia, and in the tunnel, at a much higher rate and in both directions. Arguably, the high premium paid for using the tunnel leads to its underutilisation. And the fact that the premium paid for the tunnel is not the same in the two directions leads to commuters facing asymmetric incentives when they cross from Asia to Europe and Europe to Asia. For instance, a commuter crossing Europe to Asia using the tunnel may opt for using a toll-free bridge in the opposite direction. The tunnel’s opening raises several questions on public policy: It may be that the toll revenue sharing may not be fair. But more importantly, the strait’s traffic load may not be distributed efficiently under the current toll schemes.

In this paper, we introduced a simple transportation model to explore these issues and offer guidance on public policy. Our model embeds in it the forces that have come into play after the opening of the Eurasian tunnel in Istanbul. In our model, two crossings connect the two sides of a city, one crossing standing for the Eurasian tunnel and the other for the two bridges. Commuters choose the crossings they use by minimising the total costs they incur, including the rate of toll, the cost of congestion, and the cost of diversion. The costs of congestion and diversion comprise such cost items as the excess time and fuel expended and the mental distress endured for waiting in congested traffic or taking a longer route.

The results of our analysis have several important implications for optimal toll design for Bosporus crossings. We find that an optimal premium rate for the tunnel induces the welfare-maximizing allocation. Therefore, the tunnel's premium rate, not its toll rate, should be fixed in the two directions for welfare maximisation. Under a fixed premium rate, commuters will have no systematic motive to change the crossings they use when travelling in the two directions. This will also help eliminate suspicions about the fairness of the distribution of toll revenue. We also point out the premium rate can be fixed without compromising the government's historical practice of tolling vehicles on the bridges only from Europe to Asia. The bridges can remain toll-free from Asia to Europe if the tunnel's toll rate is reduced proportionately in this direction.

Our theoretical analysis cannot provide a numerical answer to the empirical question of the magnitude of the tunnel's optimal premium rate. Yet, our results shed some light on this question. We call a crossing "overpreferred" if it would be relatively in high demand under ceteris paribus conditions (i.e., under similar toll rates and congestion levels). We find that under the welfare-maximizing allocation, the congestion level cannot be lower in the overpreferred crossing. This result hints that in our Bosporus setting, the Eurasian tunnel's toll rate should not be set so high as to divert away too many commuters leading to overcongestion on the two bridges. This is, however, in contradiction with the current practice. Therefore, in light of our findings, we recommend an amendment to the current toll schemes. In a nutshell, we recommend lowering the Eurasian tunnel's premium rate and setting the same in the two directions.

References

Anderson, S.P. et al. (1997), "Privatization and efficiency in a differentiated industry", *European Economic Review*, 41(9), 1635-1654.

Aydın, M. et al. (2019), *Türkiye Sosyal-Siyasal Eğilimler Araştırması-2018*, Kadir Has Üniversitesi.

Bös, D. (2015), *Pricing and price regulation: an economic theory for public enterprises and public utilities*, Elsevier, Advanced Textbooks in Economics Nr. 34.

Casadesus-Masanell, R. & P. Ghemawat (2006), "Dynamic mixed duopoly: A model motivated by Linux vs Windows", *Management Science*, 52(7), 1072-1084.

Cassidy, M.J. et al. (2011), "Macroscopic Fundamental Diagrams for Freeway Networks: Theory and Observation", *Transportation Research Record: Journal of the Transportation Research Board*, 2260(1), 8-15.

Condon, T. (1994), "Privatization: A Theoretical Treatment, Dieter Bös, Oxford Univ. Press, Oxford, 1991, ix + 315 pp., index, $65.00", *Journal of Comparative Economics*, 19(2), 281-285.

Cremer, H. et al. (1989), "The public firm as an instrument for regulating an oligopolistic market", *Oxford Economic Papers*, 41(2), 283-301.

Czerny, A. et al. (2014), "Hub port competition and welfare effects of strategic privatization", *Economics of Transportation*, 3(3), 211-220.

de Fraja, G. & F. Delbono (1989), "Alternative strategies of a public enterprise in oligopoly", *Oxford Economic Papers*, 41(1), 302-311.

Estrin, S. & D. de Meza (1995), "Unnatural monopoly", *Journal of Public Economics*, 57(3), 471-488.

Fujiwara, K. (2007), "Partial privatization in a differentiated mixed oligopoly", Journal of Economics, 92, 51-65.

Geroliminis, N. & C.F. Daganzo (2008), "Existence of urban-scale macroscopic fundamental diagrams: Some experimental findings", *Transportation Research Part B: Methodological*, 42(9), 759-770.

Geroliminis, N. & J. Sun (2011), "Properties of a well-defined macroscopic fundamental diagram for urban traffic", *Transportation Research Part B: Methodological*, 45(3), 605-617.

Harris, R.G. & E.G. Wiens (1980), "Government Enterprise: An Instrument for the Internal Regulation of Industry", *The Canadian Journal of Economics*, 13(1), 125-132.

Ishibashi, I. & T. Matsumura (2006), "R&D competition between public and private sectors", *European Economic Review*, 50(6), 1347-1366.

Kara Yolları Genel Müdürlüğü (2016), *2015 Trafik ve Ulaşım Bilgileri*.

Kara Yolları Genel Müdürlüğü (2019), *2018 Trafik ve Ulaşım Bilgileri*.

Mantin, B. (2012), "Airport complementarity: Private vs government ownership and welfare gravitation", *Transportation Research Part B: Methodological*, 46(3), 381-388.

Matsumura, T. & O. Kanda (2005), "Mixed Oligopoly at Free Entry Markets", *Journal of Economics*, 84, 27-48.

Matsushima, N. & T. Matsumura (2003), "Mixed oligopoly and spatial agglomeration", *Canadian Journal of Economics*, 36(1), 62-87.

Merrill, W.C. & N. Schneider (1966), "Government Firms in Oligopoly Industries: A Short-Run Analysis", *The Quarterly Journal of Economics*, 80(3), 400-412.

Nett, L. (1993), "Mixed oligopoly with homogeneous goods", *Annals of Public and Cooperative Economics*, 64(3), 367-393.

Nisan, N. et al. (2007), *Algorithmic Game Theory*, Cambridge University Press, Cambridge.

Qin, F. et al. (2017), "The welfare effects of nationalization in a mixed duopoly public transport market", *Operational Research*, 17, 593-618.

Saberi, M. & H.S. Mahmassani (2012), "Exploring Properties of Networkwide Flow-Density Relations in a Freeway Network", *Transportation Research Record: Journal of the Transportation Research Board*, 2315(1), 153-163.

Yang, H. & A. Zhang (2012), "Effects of high-speed rail and air transport competition on prices, profits and welfare", *Transportation Research Part B: Methodological*, 46(10), 1322-1333.

Appendix A: The Triangular Flow-Density Curve

In Assumption 1, we assume that the rate of congestion function φ is convex. Since Assumption 1 is critical for our results, we devote this part to its justification. First, we need to recall a few concepts from the transportation literature.

Take a reference point on the road. The *velocity* is the speed at which vehicles pass through this reference point. Velocity is a function of the traffic density, t. Let $velocity(t)$ be the velocity function.

Let $time(t)$ be the function that shows how much time it takes to traverse a given road segment. Obviously, $time(t)$ is inversely proportional to the velocity of vehicles. Thus, we can write

$$time(t) = A \cdot \frac{1}{velocity(t)}$$

where A is a constant.

In the transportation literature, *flow* is defined as the number of vehicles passing a reference point in one unit of time. The flow is proportional to the multiplication of traffic density and velocity. Thus, we can write

$$flow(t) = B \cdot t \cdot velocity(t),$$

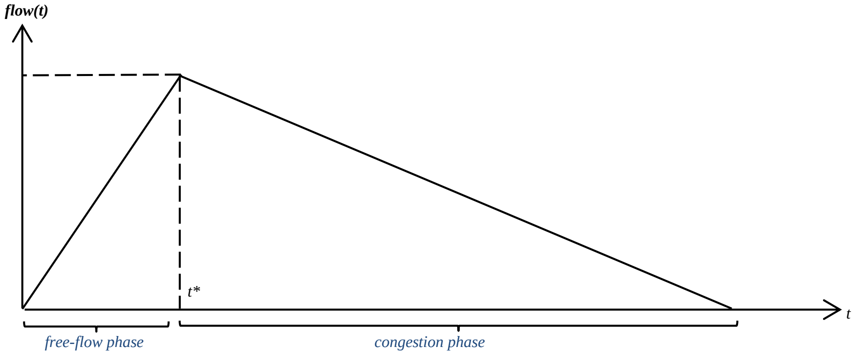
where B is a constant.

Using the above two equations, we get

$$time(t) = A \cdot B \cdot \frac{t}{flow(t)}$$

There is no direct relationship between flow and traffic density. On the one hand, the high traffic density means more vehicles are on the road, increasing the flow. On the other hand, when the traffic density is high, there is congestion, and the velocity is low, which tends to decrease the flow. The most common view of transportation literature about the relationship between flow and traffic density is that a triangular flow-density curve is the most accurate representation of real-world events. The triangular view is illustrated below in Figure 1.

Figure: 1
Triangular Flow-Density Curve



According to the triangular view, as shown in Figure 1, the flow increases linearly in traffic density t up to some level t^*. This interval is called the *free-flow phase*. Arguably, there is little traffic congestion in the free-flow phase; therefore, the flow increases linearly in traffic density.

It is called the *congestion phase* when traffic density exceeds the critical level t^*. According to the triangular view, in the congestion phase, as traffic density increases, the velocity declines fast. Consequently, the flow decreases at a constant rate, as shown in Figure 1. We can write this relationship as

$$flow(t) = D - \lambda t,$$

where $D, \lambda > 0$ are constants.

Therefore, in the congestion phase, we obtain:

$$time(t) = A \cdot B \cdot \frac{t}{flow(t)} = \frac{ABt}{D-\lambda t}$$

The first two derivatives of the function $time(t)$ are as follows:

$$\frac{dtime(t)}{dt} = \frac{ABD}{(D-\lambda t)^2} > 0, \frac{d^2time(t)}{dt^2} = \frac{2\lambda ABD}{(D-\lambda t)^3} > 0$$

Therefore, according to the widely-held triangular view, in the congestion phase, a commuter’s travel time increases convexly as traffic density increases. Consequently, if congestion cost is proportional to the travel time, the triangular view justifies our Assumption 1, that the rate of the congestion cost function is convex. Indeed, certain cost items will increase in time even faster than the linear rate, such as the costs of tardiness and commuter discomfort, which lends additional support to our Assumption 1.

Appendix B: Omitted Proofs

Proof of Proposition 5

- If $\mu(T_1, T_2) = 1$, it must be that under $\mu(T_1, T_2)$, type 2 commuters find it optimal to use crossing 1. (Type 1 commuters' choice is non-binding.) Then, we get:

$$T_1 + c + \pi \frac{1}{k} \leq T_2 \Rightarrow T_1 \leq T_2 - c - \pi \frac{1}{k}$$

- If $\mu(T_1, T_2) \in (x, 1)$, it must be that under $\mu(T_1, T_2)$, type 2 commuters face the same rate of aggregate cost at crossings 1 and 2. (Type 1 commuters' choice is non-binding.) Then, we get:

$$T_1 + c + \pi \frac{\mu(T_1,T_2)}{k} = T_2 + \pi \frac{1-\mu(T_1,T_2)}{1-k}$$

$$\Rightarrow \mu(T_1, T_2) = k - (T_1 - T_2 + c) \frac{k(1-k)}{\pi}$$

Also, for consistency, we must have:

$$\mu(T_1, T_2) \in (x, 1) \Rightarrow x < k - (T_1 - T_2 + c) \frac{k(1-k)}{\pi} < 1$$

$$\Rightarrow T_2 - c - \pi \frac{1}{k} < T_1 < T_2 - c - \pi \frac{x-k}{k(1-k)}$$

- If $\mu(T_1, T_2) = x$, it must be that under $\mu(T_1, T_2)$, type 1 commuters find it optimal to use crossing 1, and type 2 commuters find it optimal to use crossing 2. Then, we get:

$$T_1 + \pi \frac{x}{k} \leq T_2 + c + \pi \frac{1-x}{1-k} \text{ and } T_2 + \pi \frac{1-x}{1-k} \leq T_1 + c + \pi \frac{x}{k}$$

$$\Rightarrow T_2 - c - \pi \frac{x-k}{k(1-k)} \leq T_1 \leq T_2 + c - \pi \frac{x-k}{k(1-k)}$$

- If $\mu(T_1, T_2) \in (0, x)$, it must be that under $\mu(T_1, T_2)$, type 1 commuters face the same rate of aggregate cost at crossings 1 and 2. Then, we get:

$$T_1 + \pi \frac{\mu(T_1,T_2)}{k} = T_2 + c + \pi \frac{1-\mu(T_1,T_2)}{1-k}$$

$$\Rightarrow \mu(T_1, T_2) = k - (T_1 - T_2 - c) \frac{k(1-k)}{\pi}$$

Also, for consistency, we must have:

$$\mu(T_1, T_2) \in (0, x) \Rightarrow 0 < k - (T_1 - T_2 - c) \frac{k(1-k)}{\pi} < x$$

$$\Rightarrow T_2 + c - \pi \frac{x-k}{k(1-k)} < T_1 < T_2 + c + \pi \frac{1}{1-k}$$

- If $\mu(T_1, T_2) = 0$, it must be that under $\mu(T_1, T_2)$, type 1 commuters find it optimal to use crossing 1. (Type 2 commuters' choice is non-binding.) Then, we get:

$$T_2 + c + \pi \frac{1}{1-k} \leq T_1$$

This concludes our proof.

Proof of Proposition 7

Let $\Pi(T_1, T_2) = T_1\, \mu(T_1, T_2)$, where Π is operator 1's profit function. Note that for fixed T_2, the function $\mu(T_1, T_2)$ is continuous in T_1. Therefore, for fixed T_2, the function $\Pi(T_1, T_2)$ is also continuous in T_1.

Excluding the corner points of T_1 from our analysis, note that:

$$\frac{d\Pi(T_1,T_2)}{dT_1} = \mu(T_1, T_2) + T_1 \frac{d\mu(T_1,T_2)}{dT_1}$$

Hence, by Proposition 5, we get:

$$\frac{d\Pi(T_1,T_2)}{dT_1} = \begin{cases} 1 & \text{for } T_1 < B_1 \\ k - (2T_1 - T_2 + c)\frac{k(1-k)}{\pi} & \text{for } B_1 < T_1 < B_2 \\ x & \text{for } B_2 < T_1 < B_3 \\ k - (2T_1 - T_2 - c)\frac{k(1-k)}{\pi} & \text{for } B_3 < T_1 < B_4 \\ 0 & \text{for } B_4 < T_1 \end{cases}$$

Above, the values of B_1, B_2, B_3, B_4 are as specified in Proposition 5. If we set $T_2 = \frac{\pi}{1-k}$, we find the following:

- For $T_1 \in (B_1, B_2)$, we get:

$$\left.\frac{d\Pi(T_1,T_2)}{dT_1}\right|_{T_2=\frac{\pi}{1-k}} = k - \left(2T_1 - \frac{\pi}{1-k} + c\right)\frac{k(1-k)}{\pi} > 2(x-k) + c\frac{k(1-k)}{\pi}$$

Note that in the derivation of the above inequality, we used the fact that $T_1 < B_2 = -c + \pi \frac{2k-x}{k(1-k)}$.

- For $T_1 \in (B_3, B_4)$, note that:

$$\left.\frac{d\Pi(T_1,T_2)}{dT_1}\right|_{T_2=\frac{\pi}{1-k}} = 0 \Rightarrow T_1 = \frac{c}{2} + \frac{\pi}{1-k}$$

$$\left.\frac{d^2\Pi(T_1,T_2)}{dT_1^2}\right|_{T_2=\frac{\pi}{1-k}} = -2\frac{k(1-k)}{\pi} < 0$$

One can verify that $T_1 = \frac{c}{2} + \frac{\pi}{1-k} \in (B_3, B_4)$ by using Assumptions 3 and 5. (We leave this to the interested reader.)

Therefore, we obtain that for $T_2 = \frac{\pi}{1-k}$, as T_1 increases, the function $\Pi(T_1, T_2)$, which is continuous, increases up to $T_1 = \frac{c}{2} + \frac{\pi}{1-k}$, and then it first declines and then remains constant. Therefore, we obtain that for $T_2 = \frac{\pi}{1-k}$, operator 1 maximizes its profit by setting $T_1 = \frac{c}{2} + \frac{\pi}{1-k}$. Since the premium $T_1 - T_2$ is equal to $\frac{c}{2}$, by Proposition 6, we also get $\mu(T_1, T_2) = \mu^w$. This concludes our proof.