NUMERICAL DISCRETIZATION OF STOCHASTIC OSCILLATORS WITH GENERALIZED NUMERICAL INTEGRATORS

by

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In this study, we propose a numerical scheme for stochastic oscillators with additive noise obtained by the method of variation of constants formula using generalized numerical integrators. For both of the displacement and the velocity components, we show that the scheme has an order of 3/2 in one step convergence and a first order in overall convergence. Theoretical statements are supported by numerical experiments.

Key words: *stochastic oscillators, numerical schemes, difference schemes, stochastic trigonemetric integrators*

Introduction

The stochastic oscillation problem where the forcing term is driven by a Brownian Motion is widely studied. There are various application areas, including physics, biology, chemistry, economics and finance [1]. The mathematical models described by the stochastic oscillators have been deeply analyzed in the literature. The existing mathematical analysis can be classified into two main categories: linear stochastic oscillations and non-linear stochastic oscillations.

For the linear part, shows that symplecticity, linear growth of its second moment and asymptotic oscillation around zero are valid for the numerical schemes for the linear oscillation problem [2]. Analyzes the stochastic resonance in a single-well anharmonic oscillator [3]. Applies a predictor-corrector method to the numerical integration of a linear stochastic oscillation problem [4]. The simulation of stochastic oscillators have been done by locally linearized integrators in [5]. Similar linear stochastic equations are considered in [6, 7].

In terms of studies of the non-linear stochastic oscillation problems the literature is very rich. For instance, considers both quadratic and cubic non-linearities [8]. Computes the mean and the variance of underdamped harmonic oscillators both for additive and multiplicative stochastic factors [9]. The noise is driven by a Poisson stochastic process instead of a Gaussian, and the solutions of the non-linear oscillation problem is studied [10].

Another context the stochastic oscillation problems are analyzed is variable masses. For example, works on stochastic forced oscillations with a time-varying mass-spring system [11]. The mass is modelled as a random variable that is a Brownian Motion with adhesion [12].

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An important study in this field, proposes a method for the numerical discretisations of various linear stochastic oscillators [13]. In that paper, it is shown that these new schemes have the same properties with the exact solution of the model problems. However, the paper only focuses on a scalar model of a stochastic oscillator with additive noise and with a high frequency. In this article, we adopt the same approach and show the results obtained in there regardless of the frequency and in a more general settings. Independence of the magnitude of the frequency allows for a much broader area of application for the stochastic oscillation models.

In this paper, we are studying numerical analysis of a stochastic oscillator with additive noise:

$$\ddot{X}_t + \omega^2 X_t = \sigma \dot{W}_t \tag{1}$$

where ω and σ are positive constants and W_t – the standard Brownian motion process.

For the stochastic differential equation (SDE) (1), a numerical scheme based on the variation-of-constants formula is developed. By defining $Y_t = \dot{X}$ the given equation can be written as a system of first-order SDE:

$$\begin{pmatrix} \mathrm{d}X_t \\ \mathrm{d}Y_t \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix} \begin{pmatrix} X_t \\ Y_t \end{pmatrix} \mathrm{d}t + \begin{pmatrix} 0 \\ \sigma \mathrm{d}W_t \end{pmatrix} \tag{2}$$

In [13], using the variation-of-constants formula the explicit numerical scheme:

$$X_{n+1} = \cos(\omega h) X_n + \frac{1}{\omega \sin(\omega h) Y_n} + \frac{\sigma}{\omega} \sin(h\omega) \Delta W_n \tag{3a}$$

$$Y_{n+1} = -\omega \sin(\omega h) X_n + \cos(\omega h) Y_n + \sigma \cos(h\omega) \Delta W_n \tag{3b}$$

for obtaining numerical approximation for the solution of eq. (2) has been used. Here h indicates the step size of the difference scheme:

$$\Delta W_n = W_{n+1} - W_n \sim \sqrt{h} N(0,1)$$

are 1-D Gaussian random variables. In [13], it was shown that one step numerical scheme of eqs. (3a) and (3b) has an order of convergence 1/2 in both displacement and velocity. That is, it was shown that for $hw \geq c_0 > 0$, the mean square errors after one step of the numerical scheme satisfy the inequalities:

$$(E\,|\,X_1 - X_h\,|^2)^{1/2} \leq C_1 \sigma h^{1/2}$$
$$(E\,|\,Y_1 - Y_h\,|^2)^{1/2} \leq C_2 \omega \sigma h^{1/2}.$$

But in our study, for the approximate solution of eq. (2), we use general form of difference scheme of eqs. (3a) and (3b). Namely, we use the difference scheme:

$$X_{n+1} = \cos(\omega h) X_n + \frac{1}{\omega \sin(\omega h) Y_n} + \frac{\sigma}{\omega} \sin\left[\phi(h\omega)\right] \Delta W_n \tag{4a}$$

$$Y_{n+1} = -\omega \sin(\omega h) X_n + \cos(\omega h) Y_n + \sigma \cos\left[\phi(h\omega)\right] \Delta W_n \tag{4b}$$

with the genaralized integral indicator $\phi(x)$. This is reduced eqs. (3a) and (3b) if we take $\phi(x) = x$. We also prove that one step convergence in both displacement and velocity for a generalized difference scheme (4a) and (4b) has an order of convergence 3/2 without any condition on the frequency and the step size.

Then in the article [13] it is shown that the stochastic trigonometric integrators eq. (3a) has an order of convergence h. But this clearly does not make sense because a numerical scheme cannot have an order 1/2 convergence in one step and at the same time an order 1 convergence in n steps. To overcome this problem, the author put an artificial condition $h\omega \geq c_0 > 0$, *i.e.* that proof is only valid for high frequency and big step sizes. Also overall convergence of velocity has not been shown. In this study, we show that the mean square errors (MSE) for both displacement and velocity has an order of h without any condition on frequency and step sizes not only for difference scheme (3a) and (3b) but also for the generalized case difference scheme (4a) and (4b). Moreover, the numerical illustrations in the same article support our results and conclusions.

Numerical schemes for stochastic oscillators with additive noise

Now let us consider the linear stochastic oscillator with additive noise:

$$\ddot{X}_t + \omega^2 X_t = \sigma \dot{W}_t \tag{5}$$

Let us write the eq. (5) as a system of first-order Ito SDE:

$$\begin{pmatrix} \mathrm{d}X_t \\ \mathrm{d}Y_t \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix} \begin{pmatrix} X_t \\ Y_t \end{pmatrix} \mathrm{d}t + \begin{pmatrix} 0 \\ \sigma \mathrm{d}W_t \end{pmatrix} \tag{6}$$

Let us find the unique solution of the eq. (6) using the method of variation-of-constants formula. In order to do this, consider the matrix:

$$A = \begin{pmatrix} 0 & 1 \\ -\omega^2 & -v \end{pmatrix}$$

The eigenvalues of the matrix A are $\pm i\omega$ with the corresponding eigenvectors $(1, \pm i\omega)^T$, respectively. Using these we can write the matrix A n its Jordan canonical form:

$$A = \begin{pmatrix} 1 & 1 \\ i\omega & -i\omega \end{pmatrix} \begin{pmatrix} i\omega & 0 \\ 0 & -i\omega \end{pmatrix} \frac{1}{-2i\omega} \begin{pmatrix} -i\omega v & -1 \\ -i\omega & 1 \end{pmatrix}$$

Therefore, we can write the exponential matrix e^{At}:

$$\mathrm{e}^{At} = \begin{pmatrix} 1 & 1 \\ i\omega & -i\omega \end{pmatrix} \begin{pmatrix} \mathrm{e}^{i\omega t} & 0 \\ 0 & \mathrm{e}^{-i\omega t} \end{pmatrix} \begin{bmatrix} 1/2 & 1/(2i\omega) \\ 1/2 & -1/(2i\omega) \end{bmatrix} = \begin{bmatrix} \cos(\omega t) & 1/w\sin(\omega t) \\ -\omega\sin(\omega t) & \cos(\omega t) \end{bmatrix}$$

Therefore, by the variation-of-constants formula the solution of the eq. (6):

$$\begin{pmatrix} X_t \\ Y_t \end{pmatrix} = \mathrm{e}^{At} \begin{pmatrix} X_0 \\ Y_0 \end{pmatrix} + \sigma \int_0^t \mathrm{e}^{A(t-s)} \mathrm{d}W \begin{pmatrix} 0 \\ \mathrm{d}W_s \end{pmatrix}$$

Hence

$$X_t = \cos(\omega t) X_0 + \frac{1}{\omega \sin(\omega t) Y_0} + \frac{\sigma}{\omega} \int_0^t \sin\left[\omega(t-s)\right] \mathrm{d}W_s$$

$$Y_t = -\omega \sin(\omega t) X_0 + \cos(\omega t) Y_0 + \sigma \int_0^t \cos\left[\omega(t-s)\right] \mathrm{d}W_s$$

sing the fact that $e^{At}e^{As} = e^{A(t+s)}$

$$X_{n+1} = \cos(\omega h) X_n + \frac{1}{\omega \sin(\omega h) Y_n} + \frac{\sigma}{\omega} \int_{t_n}^{t_{n+1}} \sin\left[\omega\left(t_{n+1} - s\right)\right] \mathrm{d}W_s$$

$$Y_{n+1} = -\omega \sin(\omega h) X_n + \cos(\omega h) Y_n + \sigma \int_{t_n}^{t_{n+1}} \cos\left[\omega\left(t_{n+1} - s\right)\right] \mathrm{d}W_s$$

are obtained. Discretising the integrals using the generalized numerical integrators gives the following explicit numerical scheme:

$$X_{n+1} = \cos(\omega h) X_n + \frac{1}{\omega \sin(\omega h) Y_n} + \frac{\sigma}{\omega} \sin\left[\phi(\omega h)\right] \Delta W_n \tag{7a}$$

$$Y_{n+1} = -\omega \sin(\omega h) X_n + \cos(\omega h) Y_n + \sigma \cos\left[\phi(\omega h)\right] \Delta W_n \tag{7b}$$

First we will show that the energy of the numerical solution of eq. (6) has a linear growth rate.

Theorem 1. The numerical solution eqs. (7a) and (7b) of the linear stochastic oscillator eq. (6) satisfies:

$$E\left\{\frac{1}{2}\left[\left(X_n^2\right)^2 + \omega^2\left(X_n^1\right)^2\right]\right\} = \frac{1}{2}\left(y_0^2 + \omega^2 x_0^2\right) + \frac{\alpha^2}{2} t_n \tag{8}$$

for all $t_n = nh$.

Proof. Let's consider the energy equation:

$$E\left\{\frac{1}{2}\left[\left(X_{n+1}^2\right)^2 + \omega^2\left(X_{n+1}^1\right)^2\right]\right\} = E\left\{\frac{1}{2}\left(-\omega \sin(\omega h) X_n + \cos(\omega h) Y_n + \sigma \cos\left[\phi(\omega h)\right] \Delta W_n\right)^2 + \right.$$

$$\left. + \omega^2 \left(\cos(\omega h) X_n + \frac{1}{\omega \sin(\omega h) Y_n} + \frac{\sigma}{\omega} \sin\left[\phi(\omega h)\right] \Delta W_n\right)^2\right\} =$$

$$= E\left\{\frac{1}{2}\left[\left(X_n^2\right)^2 + \omega^2\left(X_n^1\right)^2\right]\right\} + \frac{\sigma^2}{2}\left(\cos^2\left[\phi(\omega h)\right] + \sin^2\left[\phi(\omega h)\right]\right) E\left[\left(\Delta Wn\right)^2\right] + 2\sigma E\left[\left(X_n^2\right)\Delta W_n\right] =$$

$$= E\left\{\frac{1}{2}\left[\left(X_n^2\right)^2 + \omega^2\left(X_n^1\right)^2\right]\right\} + \left(\frac{\sigma^2}{2}\right) h = \ldots = E\left\{\frac{1}{2}\left[\left(X_n^2\right)^2 + \omega^2\left(X_n^1\right)^2\right]\right\} + \left(\frac{\sigma^2}{2}\right)(n+1) h$$

Lemma 1. Consider the numerical solution of eq. (6) by the method (7a) and (7b). Assume that:

$$\left|\phi(\omega h) - (h - s)\omega\right| \le s\omega \text{ for any } 0 \le s \le h$$

Then the MSE after one step of the numerical scheme satisfy:

$$(E \mid X_1 - X_h \mid^2)^{1/2} \quad \le C_1 \sigma h^{3/2}$$
$$(E \mid Y_1 - Y_h \mid^2)^{1/2} \quad \le C_2 \omega \sigma h^{3/2}$$

where the constants C_1 and C_2 are independent of ω, σ, and h. Here, (X_h, Y_h) denotes the exact solution and (X_1, Y_1) denotes the numerical approximation at time h.

Proof.

$$E\,|\,X_1-X_h\,|^2=E\left[\left(\int_0^h \omega^{-1}\sigma\left\{\sin\left[\phi(\omega h)\right]-\sin\left[(h-s)\right]\omega\right\}(\mathrm{s})\,\mathrm{d}W_s\right)^2\right]$$

using the Ito isometry, we get:

$$=\omega^{-2}\sigma\int_0^h\left\{\left(\sin\left[\phi(\omega h)\right]-\sin\left[(h-s)\omega\right]\right)\right\}^2\,\mathrm{d}s$$

Now consider the function $f(x)=\sin(x)$. Then by the mean value theorem there exists an $\xi(s)$ between $\phi(\omega h)$ and $(h-s)\omega$ such that:

$$\left|\sin\left[\phi(\omega h)\right]-\sin\left[(h-s)\omega\right]\right|=\left|\cos\left[\xi(s)\omega\right]\right|\left|\phi(\omega h)-(h-s)\omega\right|-\le\left|\phi(\omega h)-(h-s)\omega\right|\le s\omega$$

so we get:

$$E\,|\,X_1-X_h\,|^2\le\omega^{-2}\sigma^2\int_0^h(s\omega)^2\,\mathrm{d}s=\frac{\sigma^2h^3}{3}$$

Therefore, the first estimate is obtained. Hence we showed that the disfference scheme (7a) has an order of convergence 3/2 in one step to the displecement part of eq. (6) The estimate for the error in the velocity component is obtained in a similar way:

$$E\,|\,Y_1-Y_h\,|^2=E\left[\left(\int_0^h\sigma\left\{\cos\left[\phi(\omega h)\right]-\cos\left[(h-s)\right]\omega\right\}\mathrm{d}W_s\right)^2\right]$$

Using the Ito isometry, we get:

$$=\sigma^2\int_0^h\left(\cos\left[\phi(\omega h)\right]-\cos\left[(h-s)\omega\right]\right)^2\,\mathrm{d}s$$

If we take the function $f(x)=\cos(h)$ then by the mean value theorem there exists an ψs between $\phi(\omega h)$ and $(h-s)\omega$ such that:

$$\left|\cos\left[\phi(\omega h)\right]-\cos\left[(h-s)\omega\right]\right|=\left|\sin\left[\psi(s)\omega\right]\right|\left|\phi(\omega h)-(h-s)\omega\right|\le\left|\phi(h\omega)-(h-s)\omega\right|\le s\omega$$

Therefore:

$$E\,|\,Y_1-Y_h\,|^2\le\sigma^2\int_0^h(s\omega)^2\,\mathrm{d}s=\frac{(\omega\sigma)^2h^3}{3}$$

This completes the proof of the *Lemma 1*. Hence, for any frequency ω as the step size h decreases, the local errors decrease with the order of 3/2 uniformly.

To show general MSE at time T, we need to obtain the following estimates.

Lemma 2. Assume that

$$\left|\phi(\omega h)-(h-s)\omega\right|\le s\omega$$

for any $0 \le s \le h$. Then the following estimates hold:

(a) $E\left|d_n^X\right| = E\left|d_n^Y\right| = 0$

(b) $E\left[\left(d_n^X\right)^2\right] = O(h3),\ E\left[\left(d_n^Y\right)^2\right] = O\left(h^3\right),\ E\left[\left|d_n^X d_n^Y\right|\right] = O\left(h^3\right),$

where

$$d_n^X = \omega^{-1}\sigma\left\{\int_{t_n}^{t_{n+1}} \sin\left[\left(t_{n+1} - s\right)\omega\right]\mathrm{d}w_s - \sin\left[\phi\left(h\omega\right)\right]\Delta W_n\right\}$$

and

$$d_n^Y = \sigma\left\{\int_{t_n}^{t_{n+1}} \cos\left[\left(t_{n+1} - s\right)\omega\right]\mathrm{d}w_s - \cos\left[\phi\left(h\omega\right)\right]\Delta W_n\right\}$$

Proof.

(a) Since the Ito stochastic integral has expectation zero, the estimates $E|d_n^X| = |d_n^Y| = 0$ follow:

(b) $E(d_n^X)^2 = \omega^{-2}\sigma^2 E\left(\int_{t_n}^{t_{n+1}} \left\{\sin\left[\left(t_{n+1} - s\right)\omega\right] - \sin\left[\phi\left(h\omega\right)\right]\right\}\mathrm{d}W_s\right)^2$

by the Ito isometry we have

$$E\left(d_n^X\right)^2 = \omega^{-2}\sigma^2 \int_{t_n}^{t_{n+1}} \left\{\sin\left[\left(t_{n+1} - s\right)\omega\right] - \sin\left[\phi\left(h\omega\right)\right]\right\}^2 \mathrm{d}s$$

by the mean value theorem

$$= \omega^{-2}\sigma^2 \int_{t_n}^{t_{n+1}} \left\{\left[\left(n+1\right)h - s\right]\omega - \phi\left(h\omega\right)\right\}^2 \left\{\cos\left[\xi\left(s\right)\omega\right]\right\}^2 \mathrm{d}s$$

for some $\xi(s)$ between $\phi(\omega h)$ and $(t_{n+1} - s)\omega$. Then:

$$\le \sigma^2 \int_{t_n}^{t_{n+1}} (nh - s)^2 \mathrm{d}s \le \frac{\sigma^2 h^3}{3}$$

since

$$\int_{t_n}^{t_{n+1}} (nh - s)^2 \mathrm{d}s = \frac{h^3}{3}$$

In the same way:

$$E(d_n^Y)^2 = \sigma^2\left(\int_{t_n}^{t_{n+1}} \left\{\cos\left[\left(t_{n+1} - s\right)\omega\right] - \cos\left[\phi\left(h\omega\right)\right]\right\}\mathrm{d}W\right)^2$$

by the Ito isometry we have:

$$=\sigma^2\int_{t_n}^{t_{n+1}}\Big(\cos\big[(t_{n+1}-s)\omega\big]-\cos\big[\phi(h\omega)\big]\Big)^2\,\mathrm{d}s$$

Again by the mean value theorem:

$$=\sigma^2\int_{t_n}^{t_{n+1}}\Big\{\big[(n+1)h-s\big]\omega\Big\}-\phi(h\omega)^2\Big\{\sin\big[\psi(s)\omega\big]\Big\}^2\,\mathrm{d}s$$

for some ψ between $\phi(\omega h)$ and $(t_{n+1}-s)\omega$. Hence, we obtain:

$$E(d_n^Y)^2\le\frac{(\omega\sigma)^2h^3}{3}$$

Let us find an estimate for $|d_n^X\,d_n^Y|$. By the fact that expectation of product of independent increments is zero, we get:

$$\left|d_n^X\,d_n^Y\right|\le\int_{t_n}^{t_{n+1}}\Big\{\sin\big[(t_{n+1}-s)\omega\big]-\sin\big[\phi(h\omega)\big]\Big\}\Big\{\cos\big[(t_{n+1}-s)\omega\big]-\sin\big[\phi(h\omega)\big]\Big\}\,\mathrm{d}s$$

By the mean value theorem:

$$\left|d_n^X\,d_n^Y\right|\le(\sigma)^2\omega\int_{t_n}^{t_{n+1}}\Big\{\big[(n+1)h-s\big]\omega-\phi(h\omega)\Big\}^2\Big|\sin\big[\psi(s)\omega\big]\Big|\Big|\cos\big[\xi(s)\omega\big]\Big|\,\mathrm{d}s$$

for some $\psi(s)$and for some $\xi(s)$between $\phi(\omega h)$ and $(t_{n+1}-s)\omega$. Then:

$$\left|d_n^X\,d_n^Y\right|\le\frac{\sigma^2\omega h^3}{3}$$

The proof is completed.

We now consider the global mean-square error of the stochastic exponential integrators eqs. (7a) and (7b).

Theorem 2. Consider the numerical solution of eq. (6), by the method eqs. (7a) and (7b). Assume that:

$$\Big|\phi(\omega h)-(h-s)\omega\Big|\le s\omega,\ \text{ for any }\ 0\le s\le h$$

Then the mean-square errors of the numerical scheme (7a) and (7b) satisfy:

$$\text{(a)}\ \Big(E\,|\,X_n-X_{t_n}\,|^2\Big)^{1/2}\le\left(\frac{T}{3}\right)^{1/2}\left(1+\frac{1}{\omega}\right)^{1/2}\sigma h$$

$$\text{(b)}\ \Big(E\,|\,Y_n-Y_{t_n}\,|^2\Big)^{1/2}\le\left(\frac{T}{3}\right)^{1/2}\big[\omega(\omega+1)\big]^{1/2}\sigma h$$

for $nh\le T$.

Proof. The recursive relation for the solution of linear part:

$$\begin{pmatrix}X_{t_{n+1}}\\Y_{t_{n+1}}\end{pmatrix}=\mathrm{e}^{Ah}\begin{pmatrix}X_{t_n}\\Y_{t_n}\end{pmatrix}+\int_{t_n}^{t_{n+1}}\mathrm{e}^{A(t_{n+1}-s)}\begin{pmatrix}0\\\sigma\dot{W}_s\end{pmatrix}\mathrm{d}s$$

Using the eqs. (7a) and (7b) we have:

$$E_{n+1} = \mathrm{e}^{Ah} E_n + d_n$$

where

$$E_n = \begin{pmatrix} e_n^X \\ e_n^Y \end{pmatrix} = \begin{pmatrix} X_{t_n} - X_n \\ Y_{t_n} - Y_n \end{pmatrix} \text{ and } d_n = \begin{pmatrix} d_n^X \\ d_n^Y \end{pmatrix}$$

Using the mathematical induction we obtain:

$$E_{n+1} = \mathrm{e}^{A(n+1)h} E_0 + \sum_{j=0}^{n} \mathrm{e}^{A(n-j)h} d_n = \sum_{j=0}^{n} \mathrm{e}^{A(n-j)h} d_j$$

where $E_0 = \vec{0}$. Hence:

$$E\left[\left(\mathrm{e}_{n+1}^X\right)\right]^2 = E\left[\sum_{j=0}^{n}\left\{\cos\left[(n-j)h\omega\right]d_j^X + w^{-1}\sin\left[(n-j)h\omega\right]d_j^Y\right\}\right]^2 =$$

$$= E\sum_{j=0}^{n}\sum_{i=0}^{n}\left\{\cos\left[(n-j)h\right]d_j^X + \omega^{-1}\sin\left[(n-j)h\right]d_j^Y\right\}\left\{\cos\left[(n-i)h\right]d_i^X + \omega^{-1}\sin\left[(n-i)h\right]d_i^Y\right\}$$

Since expectation of product of independent increments is zero, we have:

$$= \sum_{j=0}^{n}\left(\left\{\cos\left[(n-j)h\omega\right]\right\}^2 E\left[\left(d_j^X\right)^2\right] + \left\{\omega^{-1}\sin\left[(n-j)h\omega\right]\right\}^2 E\left[\left(d_j^Y\right)^2\right]\right) +$$

$$+2\omega^{-1}\sum_{j=0}^{n}\left\{\cos\left[(n-j)h\omega\right]\sin\left[(n-j)h\omega\right]E\left(d_j^X d_j^Y\right)\right\} \leq$$

$$\leq \sum_{j=0}^{n}\left(\left\{\cos\left[(n-j)h\omega\right]\right\}^2 \sigma^2 h^3/3 + \left\{\omega^{-1}\sin\left[(n-j)h\omega\right]\right\}^2 w^2\sigma^2 h^3/3\right) +$$

$$+2\omega^{-1}\sum_{j=0}^{n}\cos\left[(n-j)h\omega\right]\sin\left[(n-j)h\omega\right]\omega\sigma^2 h^3/3 \leq$$

$$\leq \sigma^2\sum_{j=0}^{n}\left\{\cos\left[(n-j)h\omega\right] + \omega^{-1}\sin\left[(n-j)h\omega\right]\right\}^2 h^3/3 \leq \frac{1}{3}\left(1+\frac{1}{\omega}\right)\sigma^2 T h^2$$

Similarly:

$$E\left[\left(\mathrm{e}_{n+1}^Y\right)^2\right] = E\left(\sum_{j=0}^{n}\left\{-\omega\sin\left[(n-j)h\omega\right]d_j^X + \cos\left[(n-j)h\omega\right]d_j^Y\right\}\right)^2 =$$

$$= \sum_{j=0}^{n}\left(\omega\left\{\sin\left[(n-j)h\omega\right]\right\}^2 E\left[\left(d_j^X\right)^2\right] + \left\{\cos\sin\left[(n-j)h\omega\right]\right\}^2 E\left[\left(d_j^Y\right)^2\right]\right) +$$

$$+2w\sum_{j=0}^{n}\left\{\sin\left[\left(n-j\right)h\omega\right]\right\}\cos\left[\left(n-j\right)h\omega\right]E\left(d_j^X d_j^Y\right)\le$$

$$\le\sum_{j=0}^{n}\left(\left\{(\omega\sin\left[\left(n-j\right)h\omega\right]\right\}^2\sigma^2h^3/3+\left\{(\cos\left[\left(n-j\right)h\omega\right]\right\}^2\omega^2\sigma^2h^3/3\right)+$$

$$+2w\sum_{j=0}^{n}\cos\left[\left(n-j\right)h\omega\right]\sin\left[\left(n-j\right)h\omega\right]w\sigma^2h^3/3\le$$

$$\le(\omega\sigma)^2\sum_{j=0}^{n}\left\{\cos\left[\left(n-j\right)h\omega\right]+\omega^{-1}\sin\left[\left(n-j\right)h\omega\right]\right\}^2h^3/3\le\frac{1}{3}\left(1+\frac{1}{\omega}\right)\left(\omega\sigma\right)^2Th^2$$

This completes the proof of the *Theorem 2*.

Note that, the estimates obtained in *Theorem 2*. show that both the displacement and the velocity component of the difference scheme have better estimates for $\omega \geq 1$ rather than $0 < \omega < 1$. However, this does not affect the convergence rate since ω is an arbitrary but fixed constant. Hence, the mentioned numerical scheme still has a first order convergence in both the displacement and the velocity components.

Numerical results

For the comparison of the numeric solution of the difference equation and the analytical solution of the differential equation, the error terms are computed by the following formulation for the position component X and the velocity component Y:

$$E_h^X=\frac{1}{N_{\mathrm{sim}}}\left[\sum_{j=1}^{N_{\mathrm{sim}}}\left(X_n-X_{t_n}\right)^2\right]^{1/2} \tag{9}$$

$$E_h^Y=\frac{1}{N_{\mathrm{sim}}}\left[\sum_{j=1}^{N_{\mathrm{sim}}}\left(Y_n-Y_{t_n}\right)^2\right]^{1/2} \tag{10}$$

Maintaining the same notation that has been used in section *Numerical schemes for stochastic oscillators with additive noise*, we represent the analytical solution of the system of eq. (6) by X_{tn} for the position factor and Y_{tn} for the velocity factor. The numerical solutions of the problem based on the eqs. (7a) and (7b) are denoted by X_n and Y_n. The error terms are recorded for various values of N, *i.e.* the number of steps in time. The results are shown in the tab. 1 for $N = 2$, $N = 4$, $N = 8$,..., $N = 512$, and for various values of ω changing from 0.1-0.5. In all of these numerical experiments the number of simulations N_{sim} is kept constant at 1000000. Hence, each numerical problem has been solved based on 1000000 different sample paths for the process of standard Brownian motion, W_t.

Although the convergence is observed for all the values of ω, it is clear that the convergence is much faster when the ω is bigger. Especially, when ω is bigger than 10 the numerical solution converges to the analytical solution even for very large step sizes. Here, the initial conditions used are $X(0) = 0.02$, and $Y(0) = 0$. The parameter σ is chosen to be 1 in all the numerical exercises. In order to check the rate of convergence, we plot the MSE in X in fig. 1

Table 1. Comparison of the errors for the approximate solution of problem for the position factor *X*

Number of steps, ω	$\omega = 0.1$	$\omega = 0.5$	$\omega = 1$	$\omega = 10$	$\omega = 50$	$\omega = 100$	$\omega = 150$
$N = 2$	0.0014	0.0017	0.0008	0.0000	0.0000	0.0000	0.0000
$N = 4$	0.0015	0.0004	0.0010	0.0001	0.0000	0.0000	0.0000
$N = 8$	0.0030	0.0020	0.0009	0.0001	0.0000	0.0000	0.0000
$N = 16$	0.0007	0.0006	0.0012	0.0001	0.0000	0.0000	0.0000
$N = 32$	0.0010	0.0002	0.0001	0.0001	0.0000	0.0000	0.0000
$N = 64$	0.0010	0.0002	0.0001	0.0001	0.0000	0.0000	0.0000
$N = 128$	0.0010	0.0003	0.0014	0.0001	0.0000	0.0000	0.0000
$N = 256$	0.0006	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000
$N = 512$	0.0006	0.0009	0.0005	0.0001	0.0000	0.0000	0.0000

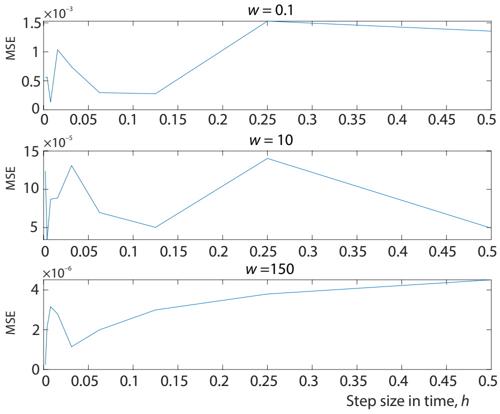

Figure 1. The MSE of the numerical solution for various values of ω at the specific time point $t = 1$ based on 1000000 number of Monte-Carlo simulations

for some values of ω. We picked on small value $\omega = 0.1$, one moderate value $\omega = 10$, and one large value $\omega = 150$. Although it is not very smooth, we observe almost a linear behavior in error terms which is in line with the order h convergence rate proven in *Theorem 1*.

We repeat a similar convergence analysis for the velocity component Y. The results are reported in tab. 2. Two main findings are the convergence is slightly slower compared to the convergence in the position component X and the error is more or less independent of the choice of ω. Almost linear convergence is still observed in fig. 2, which confirms the result obtained in *Theorem 2*.

Figures 3 and 4 in the article [13] are in line with our results.

Table 2. Comparison of the errors for the approximate solution of problem for the velocity component *Y*

Number of steps, ω	$\omega = 0.1$	$\omega = 0.5$	$\omega = 1$	$\omega = 10$	$\omega = 50$	$\omega = 100$	$\omega = 150$
N = 2	0.0016	0.0019	0.0017	0.0004	0.0002	0.0013	0.0003
N = 4	0.0014	0.0002	0.0018	0.0002	0.0018	0.0007	0.0012
N = 8	0.0040	0.0030	0.0008	0.0004	0.0013	0.0008	0.0010
N = 16	0.0001	0.0005	0.0017	0.0002	0.0006	0.0011	0.0006
N = 32	0.0009	0.0004	0.0016	0.0006	0.0007	0.0012	0.0010
N = 64	0.0000	0.0001	0.0003	0.0001	0.0005	0.0005	0.0003
N = 128	0.0000	0.0000	0.0029	0.0001	0.0014	0.0004	0.0010
N = 256	0.0006	0.0000	0.0003	0.0003	0.0000	0.0008	0.0010
N = 512	0.0016	0.0009	0.0001	0.0001	0.0010	0.0009	0.0007

Conclusion

In this article, we proved the convergence of difference schemes for the stochastic oscillator with additive noise for the generalized numerical integrators. Without any condition of high frequency and big step sizes, the difference schemes proposed are proved to have an order of convergence 3/2 after one step of the difference scheme and an order of convergence 1 globally in both displacement and velocity components.

Numerical analysis of the suggested difference scheme has been executed. The numerical results almost confirm the order of convergence obtained. The convergence for the position component X is better for large frequencies, however the frequency is irrelevant for the convergence of the velocity component Y.

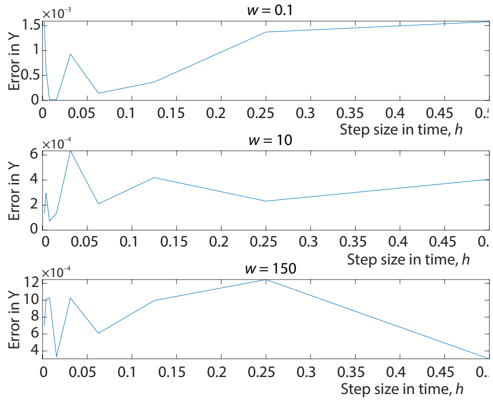

Figure 2. The MSE of the numerical solution of the velocity component Y for various values of ω at the specific time point t = 1 based on 1000000 number of Monte-Carlo simulations

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