Fractional Calculus and Applied Analysis (2024) 27:22–63
https://doi.org/10.1007/s13540-023-00236-2

ORIGINAL PAPER

Lump solutions of the fractional Kadomtsev–Petviashvili equation

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Received: 30 May 2023 / Revised: 5 December 2023 / Accepted: 27 December 2023 /
Published online: 10 January 2024

Abstract

Of concern is the fractional Kadomtsev–Petviashvili (fKP) equation and its lump solution. As in the classical Kadomtsev–Petviashvili equation, the fKP equation comes in two versions: fKP-I (strong surface tension case) and fKP-II (weak surface tension case). We prove the existence of nontrivial lump solutions for the fKP-I equation in the energy subcritical case $\alpha > \frac{4}{5}$ by means of variational methods. It is already known that there exist neither nontrivial lump solutions belonging to the energy space for the fKP-II equation [9] nor for the fKP-I when $\alpha \leq \frac{4}{5}$ [26]. Furthermore, we show that for any $\alpha > \frac{4}{5}$ lump solutions for the fKP-I equation are smooth and decay quadratically at infinity. Numerical experiments are performed for the existence of lump solutions and their decay. Moreover, numerically, we observe cross-sectional symmetry of lump solutions for the fKP-I equation.

Keywords Fractional Kadomtsev-Petviashvili equation (primary) · Existence of lump solutions · Decay of lump solutions · Petviashvili iteration

Mathematics Subject Classification 35Q35 (primary) · 35C07 · 35S30 · 35B65 · 35B06 · 35B40 · 45K05

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1 Introduction

The present paper is devoted to the study of fully localized solitary solutions (also known as *lump solutions*) of the fractional Kadomtsev–Petviashvili (fKP) equation

$$u_t + uu_x - \mathrm{D}_x^\alpha u_x + \sigma \partial_x^{-1} u_{yy} = 0. \tag{1.1}$$

Here the real function $u = u(t, x, y)$ depends on the spatial variable $(x, y) \in \mathbb{R}^2$ and the temporal variable $t \in \mathbb{R}_+$. The linear operator D_x^α denotes the Riesz potential of order $\alpha \in \mathbb{R}$ in x-direction, which is defined by multiplication with $|\cdot|^\alpha$ on the frequency space, that is

$$\mathcal{F}(\mathrm{D}_x^\alpha f)(t, \xi_1, \xi_2) = |\xi_1|^\alpha \hat{f}(t, \xi_1, \xi_2),$$

where the operator $\mathcal{F}$ denotes the extension to the space of tempered distributions $\mathcal{S}'(\mathbb{R}^n)$ of the Fourier transform $\mathcal{F}(f)(\xi) := \int_{\mathbb{R}^n} f(x) e^{-\mathrm{i}\xi x}\, dx$ on the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ with inverse $\mathcal{F}^{-1}(f) := \frac{1}{2\pi}\mathcal{F}(f)(-\cdot)$. We also write $\hat{f} := \mathcal{F}(f)$. The operator ∂_x^{-1} is defined as a Fourier multiplier operator on the x-variable as $\mathcal{F}(\partial_x^{-1} f)(t, \xi_1, \xi_2) = \frac{1}{\mathrm{i}\xi_1} \hat{f}(t, \xi_1, \xi_2)$. In the case $\alpha = 2$ equation (1.1) becomes the classical Kadomtsev–Petviashvili (KP) equation which was introduced by Kadomtsev & Petviashvili [19] as a weakly two-dimensional extension of the celebrated Korteweg–de Vries (KdV) equation,

$$u_t + uu_x + u_{xxx} = 0,$$

which is a spatially one-dimensional equation appearing in the context of small-amplitude shallow water-wave model equations. The KP equation comes in two versions: For $\sigma = -1$ it is called KP-I and for $\sigma = 1$ it is called KP-II. Roughly speaking, the KP-I equation represents the case of strong surface tension, while the KP-II equation appears as a model equation for weak surface tension. Analogously to the classical case, the fKP equation is a two-dimensional extension of the fractional Korteweg–de Vries (fKdV) equation

$$u_t + uu_x - \mathrm{D}_x^\alpha u_x = 0$$

and (1.1) is referred to as the fKP-I equation when $\sigma = -1$ and as the fKP-II equation when $\sigma = 1$. Notice that for $\alpha = 1$ in (1.1) we recover the KP-version of the Benjamin–Ono equation. During the last decade there has been a growing interest in fractional regimes such as the fKdV or the fKP equation (see for example [1, 4, 14–16, 20, 21, 23, 26, 27, 29, 31, 32, 34, 38] and the references therein). Even though most of these equations are not derived by asymptotic expansions from governing equations in fluid dynamics they can be thought of as dispersive corrections.

Formally, the fKP equation does not only conserve the L^2–norm

$$M(u) = \int_{\mathbb{R}^2} u^2 \,\mathrm{d}(x, y),$$

but also the energy

$$E_\alpha(u) := \int_{\mathbb{R}^2} \left(\frac{1}{2}(\mathrm{D}_x^{\frac{\alpha}{2}} u)^2 - \tfrac{1}{6}u^3 - \tfrac{1}{2}\sigma(\partial_x^{-1} u_y)^2 \right) \mathrm{d}(x, y).$$

Notice that the corresponding energy space

$$X_{\frac{\alpha}{2}}(\mathbb{R}^2) := \{u \in L^2(\mathbb{R}^2) \mid \mathrm{D}_x^{\frac{\alpha}{2}} u, \partial_x^{-1} u_y \in L^2(\mathbb{R}^2)\}$$

equipped with the norm

$$||\phi||^2_{\frac{\alpha}{2}} := \|\phi\|^2_{L^2(\mathbb{R}^2)} + \left\| \mathrm{D}_x^{\frac{\alpha}{2}} \phi \right\|^2_{L^2(\mathbb{R}^2)} + \left\| \partial_x^{-1} \partial_y \phi \right\|^2_{L^2(\mathbb{R}^2)},$$

includes a zero-mass constraint with respect to x. We refer to [26] for derivation issues and well-posedness results for the Cauchy problem associated with (1.1). The fKP equation is invariant under the scaling

$$u_\lambda(t, x, y) = \lambda^\alpha u(\lambda^{\alpha+1} t, \lambda x, \lambda^{\frac{\alpha+2}{2}} y),$$

and $\|u_\lambda\|_{L^2} = \lambda^{\frac{3\alpha-4}{4}} \|u\|_{L^2}$. Thus, $\alpha = \frac{4}{3}$ is the L^2-*critical exponent* for the fKP equation. The ranges $\alpha > \frac{4}{3}$ and $\alpha < \frac{4}{3}$ are called *sub*- and *supercritical*, respectively. Due to the embedding $X_{\frac{\alpha}{2}} \subset L^3(\mathbb{R}^2)$ for $\alpha \geq \frac{4}{5}$ (cf. [26, Lemma 1.1]), we call $\alpha = \frac{4}{5}$ the *energy critical exponent* for the fKP equation.

A traveling-wave solution $u(t, x, y) = \phi(x - ct, y)$ of the fKP equation propagating in x-direction with wave speed $c > 0$, satisfies the steady equation

$$-c\phi + \frac{1}{2}\phi^2 - \mathrm{D}_x^\alpha \phi + \sigma \partial_x^{-2} \phi_{yy} = 0. \tag{1.2}$$

Lump solutions are traveling-wave solutions decaying to 0 as $|(x, y)| \to \infty$.

1.1 Main results

Our aim is to study the existence and spatial decay of lump solutions for the fKP equation. Since it is known [9, 26] that the fKP-II equation for any α as well as the fKP-I equation for $\alpha \leq \frac{4}{5}$ do not admit any lump solutions in $X_{\frac{\alpha}{2}} \cap L^3(\mathbb{R}^2)$, the study of this paper is concerned with traveling waves for the fKP-I equation for $\alpha > \frac{4}{5}$. We prove the following two main theorems. Moreover, we study lump solutions and some of their properties numerically.

Theorem 1 *(Existence of lump solutions) For any $\frac{4}{5} < \alpha$ there exists a lump solution $\phi \in X_{\frac{\alpha}{2}}$ of (1.2) with $\sigma = -1$.*

Theorem 2 *(Decay of lump solutions) Any lump solution $\phi \in X_{\frac{\alpha}{2}}$ of (1.2) with $\sigma = -1$ is smooth and satisfies*

$$\varrho^2 \phi \in L^\infty(\mathbb{R}^2), \qquad \textit{where} \quad \varrho^2(x,y) = x^2 + y^2.$$

The classical KP-I equation possesses an explicit lump solution of the form

$$\phi_e(x-ct, y) = 8c \frac{1 - \frac{c}{3}(x-ct)^2 + \frac{c^2}{3} y^2}{\left(1 + \frac{c}{3}(x-ct)^2 + \frac{c^2}{3} y^2\right)^2}. \tag{1.3}$$

We would like to point out that De Bouard & Saut [9] studied the existence of of lump solutions for the generalized KP-I equation

$$(u_t + u^p u_x - u_{xxx})_x - u_{yy} = 0 \tag{1.4}$$

where $p = m/n \geq 1$, m, n relatively prime and n odd. Furthermore, in their continuation paper [10], de Bouard & Saut investigated the symmetry and decay of lump solutions for (1.4) and showed that for all $p \geq 1$ the decay is quadratic. Our studies follow a similar approach as in [9, 10]. However, special attention needs to be given to the nonlocal operator D_x^α. While many proofs can be adapted with a bit more technical effort due to the nonlocal operator, the result on decay of lump solutions in the supercritical case $\frac{4}{5} < \alpha < \frac{4}{3}$ (which includes the Benjamin–Ono KP version for $\alpha = 1$) needs a modified approach, since in the supercritical case the symbol of an operator related to the linear dispersion is no longer L^2-integrable.

On the existence result: We give a brief outline of the existence proof for lump solutions of (1.4) in [9] by variational methods, since we will be using the same strategy to prove existence of lump solutions of the fKP-I equation (1.1). First consider the constrained minimization problem

$$I_\mu = \inf \left\{ \|\phi\|_Y^2 : \phi \in Y, \int_{\mathbb{R}^2} \phi^{p+2} = \mu \right\}$$

for $\mu > 0$ fixed, where Y is the closure of $\partial_x(C_0^\infty(\mathbb{R}^2))$ (the space of functions of the form $\partial_x \varphi$ with $\varphi \in C_0^\infty(\mathbb{R}^2)$) with respect to the norm

$$\|\partial_x \varphi\|_Y^2 = \|\nabla \varphi\|_{L^2(\mathbb{R}^2)}^2 + \left\| \partial_x^2 \varphi \right\|_{L^2(\mathbb{R}^2)}^2.$$

Via the Lagrange multiplier principle one finds (after rescaling) that solutions of the constrained minimization problem I_μ are lump solutions of (1.4). The task is then to prove existence of solutions of I_μ and this is achieved using the concentration-compactness theorem (cf. Theorem 3). The variational formulation associated with I_μ has several good properties. The functional being minimized is just the norm of the space Y. It is therefore immediate that it is coercive, bounded from below and weakly lower semi-continuous; properties which are all advantageous in the context of minimization problems, see [39, Theorem 1.2]. Furthermore, since the norm is homogeneous, it is easily shown that I_μ is subadditive as a function of μ and this property is essential in proving that the dichotomy scenario in the concentration-compactness theorem does not occur.

We prove Theorem 1 by extending the strategy of [9], outlined above, to the fractional case. Generally speaking, the fractional derivative and the fact that we are allowing for weak dispersion makes the proof of Theorem 1 more technical than its classical local counterpart ($\alpha = 2$). A key ingredient in the proof is the anisotropic Sobolev inequality [26, Lemma 1.1] (see also Proposition 1 (ii)), which in particular says that for $\frac{4}{5} \leq \alpha$, the space $X_{\frac{\alpha}{2}}$ is continuously embedded in $L^3(\mathbb{R}^2)$. This result is what determines the values of α for which we can prove existence of solitary waves. In fact, for $\alpha \leq \frac{4}{5}$ there exist no nontrivial lump solutions of for the fKP-I equation in $X_{\frac{\alpha}{2}} \cap L^3(\mathbb{R}^2)$ [26, Proposition 1.2].

We would like to mention that there are several existence results on lump solutions using variational approaches for other two-dimensional equations. The full water-wave problem admits lump solutions both for strong [5, 18] and weak [6] surface tension. In the strong surface tension case the lump solutions can be approximated by rescalings of KP-I lumps, while in the weak surface tension case the lump solutions can be approximated by rescalings of Davey-Stewartson type solitary waves. The full dispersion KP (FDKP) equation was introduced in [25, chapter 8] as a model for weakly transversal three dimensional water-waves which preserves the dispersion relation of the full water-wave problem. A comparison of the fDKP equation with the KP equation for the propagation of water waves is given in [28]. Just as for the classical and fractional KP equation, the FDKP equation can be considered for both strong (FDKP-I) and weak (FDKP-II) surface tension. In [11] it was shown that the FDKP-I equation admits lump solutions and later on in [12] it was shown that also the FDKP-II equation possesses lump solutions. This is in contrast to the fKP-II equation, which does not admit any lump solutions [26]. Just like for the full water-wave problem, in the strong surface tension case the lump solutions can be approximated by rescalings of KP-I lumps, while in the weak surface tension case the lump solutions can be approximated by rescalings of Davey–Stewartson type solitary-waves.

On the decay result: The proof of Theorem 2 on the decay properties of lump solutions is closely related to that of [3, Theorem 3.1.2] and [10, Theorem 4.1]. The steady equation (1.2) can be rewritten as a convolution equation of the form

$$\phi = \frac{1}{2} K_\alpha * \phi^2, \qquad \hat{K}_\alpha(\xi_1, \xi_2) = m_\alpha(\xi_1, \xi_2),$$

where the symbol m_α is given by $m_\alpha(\xi_1, \xi_2) = \frac{\xi_1^2}{|\xi|^2 + \xi_1^{\alpha+2}}$.

Remark 1 An immediate consequence of the discontinuity of the symbol m_α at the origin is that any nontrivial, continuous lump solution of (1.2) decays *at most* quadratically. Let us assume for a contradiction that ϕ is a nontrivial, continuous lump solution, which decays at infinity as $|\cdot|^{-\delta}$ for some $\delta > 2$. Then $\phi \in L^1(\mathbb{R}^2)$, which implies that the Fourier transformation of ϕ is continuous. But $\hat{\phi} = \frac{1}{2} m_\alpha \hat{\phi^2}$ cannot be continuous at the origin, since $\hat{\phi^2}(0,0) > 0$ and m_α is discontinuous at the origin. We conclude that the singularity of the symbol m_α induced by the transverse direction forces the decay of any nontrivial, continuous lump solution to be at most quadratic.

Remark 2 In view of Remark 1 the decay rate in Theorem 2 is optimal.

The idea is to study the kernel function K_α and to show that it has exactly quadratic decay at infinity (independent of α). Then the decay properties of K_α are used to show that also ϕ decays quadratically at infinity.

On the numerics: We conduct numerical experiments to observe the lump solutions and some of their properties. For this purpose, we generate the solutions numerically by using Petviashvili iteration method. The method was proposed first by Petviashvili [37] to compute the lump solutions of the KP-I equation. The convergence of the method for the KP-equation was later discussed in [35] and now it is widely used to numerically evaluate traveling wave solutions of evolution equations (see for example [2, 33, 36] and the references therein).

Applying the Fourier transform to (1.2) with respect to the space variables (x, y) we obtain

$$c\widehat{\phi} - \frac{1}{2}\widehat{\phi^2} + |\xi_1|^\alpha \widehat{\phi} + \frac{\xi_2^2}{\xi_1^2}\widehat{\phi} = 0. \tag{1.5}$$

An iterative algorithm for the equation (1.5) can be proposed as

$$\widehat{\phi}_{n+1}(\xi_1, \xi_2) = \frac{\widehat{\phi_n^2}(\xi_1, \xi_2)}{2(c + \frac{\xi_2^2}{\xi_1^2} + |\xi_1|^\alpha)}, \quad n = 1, 2, \ldots, \tag{1.6}$$

where ϕ_n is the n^{th} iteration of the numerical solution. Since (1.6) is generally divergent, the Petviashvili iteration is given as

$$\widehat{\phi}_{n+1}(\xi_1, \xi_2) = \frac{(M_n)^\nu}{2(c + \frac{\xi_2^2}{\xi_1^2} + |\xi_1|^\alpha)}\widehat{\phi_n^2}(\xi_1, \xi_2), \quad n = 1, 2, \ldots, \tag{1.7}$$

by introducing the stabilizing factor

$$M_n = \frac{\int_{\mathbb{R}^2} 2(c + \frac{\xi_2^2}{\xi_1^2} + |\xi_1|^\alpha)\, (\widehat{\phi_n})^2 \mathrm{d}(\xi_1, \xi_2)}{\int_{\mathbb{R}^2} \widehat{\phi_n^2}\, \widehat{\phi_n}\, \mathrm{d}(\xi_1, \xi_2)}.$$

Here the free parameter ν is chosen as 2 for the fastest convergence. To evaluate the term $1/\xi_1^2$ for $\xi_1 = 0$, we regularize it as $1/(\xi_1 + i\lambda)^2$, where $\lambda = 2.2 \times 10^{-16}$ as in [22, 24]. We control the iterative process by the error between two consecutive iterations

$$\text{error}(n) = \|\phi_n - \phi_{n-1}\|_\infty, \quad n = 1, 2, \ldots,$$

by the stabilization factor error $|1 - M_n|$, and the residual error

$$Res(n) = \|\mathcal{S}\phi_n\|_\infty, \quad n = 1, 2, \ldots$$

where

$$\mathcal{S} = \left(-c\phi + \frac{1}{2}\phi^2 - \mathrm{D}_x^\alpha \phi\right)_{xx} - \phi_{yy}.$$

We make sure that the errors are of order less than 10^{-5}. In addition, we control the decay of Fourier coefficients $\hat{\phi}(\xi_1, \xi_2)$ in the numerical experiments.

1.2 Notation and organization of the paper

We first introduce a notation, which is frequently used in the sequel. Let f and g be two positive functions. We write $f \lesssim g$ ($f \gtrsim g$) if there exists a constant $c > 0$ such that $f \le cg$ ($f \ge cg$). Moreover, we use the notation $f \eqsim g$ whenever $f \lesssim g$ and $f \gtrsim g$.

We conclude the introduction by the organization of the paper: In Section 2 we prove existence of lump solutions for the fKP-I equation (Theorem 1) via a variational approach. We also present numerically generated lump solutions and observe the cross-sectional symmetry of the solutions numerically. Section 3 is devoted to the proof of Theorem 2, which relies upon a careful study of the decay and regularity of the kernel function K_α. The appendix contains some technical results which are needed for the analysis in Section 3.

2 Existence of solitary wave solutions

We consider the (rescaled) traveling wave fKP-I equation:

$$\phi + \mathrm{D}_x^\alpha \phi + \partial_x^{-2}\partial_y^2 \phi - \frac{\phi^2}{2} = 0. \tag{2.1}$$

Equation (2.1) can be realized as a constrained minimization problem. Indeed, let

$$\mathcal{L}(\phi) = \frac{1}{2}\int_{\mathbb{R}^2} \left(\phi^2 + (\mathrm{D}_x^{\frac{\alpha}{2}}\phi)^2 + (\partial_x^{-1}\partial_y\phi)^2\right) \mathrm{d}(x, y),\ \mathcal{N}(\phi) = \frac{1}{6}\int_{\mathbb{R}^2} \phi^3\, \mathrm{d}(x, y),$$

which we study in the space $X_{\frac{\alpha}{2}}$ and consider the constrained minimization problem

$$I_\mu = \inf\{\mathcal{L}(\phi) \colon \phi \in X_{\frac{\alpha}{2}},\ \mathcal{N}(\phi) = \mu\}. \tag{2.2}$$

In order to find nontrivial solutions we assume that $\mu \neq 0$ and without loss of generality we may further assume that $\mu > 0$.

Let ϕ be a solution of (2.2). Then there exists a Lagrange multiplier $\lambda \in \mathbb{R}$ such that

$$d\mathcal{L}(\phi) - \lambda d\mathcal{N}(\phi) = 0. \tag{2.3}$$

Since

$$d\mathcal{L}(\phi) = \phi + D_x^\alpha \phi + \partial_x^{-2}\partial_y^2\phi, \qquad d\mathcal{N}(\phi) = \frac{1}{2}\phi^2,$$

equation (2.3) becomes

$$\phi + D_x^\alpha \phi + \partial_x^{-2}\partial_y^2\phi - \lambda\frac{\phi^2}{2} = 0.$$

By rescaling $\phi(x, y) = \lambda^{-1}\tilde{\phi}(x, y)$, we find that $\tilde{\phi}$ satisfies the equation

$$\tilde{\phi} + D_x^\alpha \tilde{\phi} + \partial_x^{-2}\partial_y^2\tilde{\phi} - \frac{\tilde{\phi}^2}{2} = 0,$$

which is (2.1). Therefore, in order to prove the existence of the solutions of equation (2.1), we will prove existence of solutions of the constrained minimization problem (2.2).

In the sequel, let us fix $\mu > 0$ (this will ensure that $I_\mu > 0$, see Corollary 1) and let $\{\phi_n\}_{n\in\mathbb{N}} \subset X_{\frac{\alpha}{2}}$ be a minimizing sequence such that $\mathcal{N}(\phi_n) = \mu$ and $\lim_{n\to\infty} \mathcal{L}(\phi_n) = I_\mu$. We aim to show that there exists a subsequence (not relabeled) of $\{\phi_n\}_{n\in\mathbb{N}}$, which converges to a function $\phi \in X_{\frac{\alpha}{2}}$ satisfying $\mathcal{L}(\phi) = I_\mu$ and $\mathcal{N}(\phi) = \mu$.

Let us set

$$e_n = \frac{1}{2}\left(\phi_n^2 + (D_x^{\frac{\alpha}{2}}\phi_n)^2 + (\partial_x^{-1}\partial_y\phi_n)^2\right)$$

and note that

$$\mathcal{L}(\phi_n) = \int_{\mathbb{R}^2} e_n \, d(x, y).$$

We will use the following version of the concentration–compactness theorem for the sequence $\{e_n\}_{n\in\mathbb{N}}$ and show that the concentration scenario occurs. This is then used to construct a convergent subsequence of $\{\phi_n\}_{n\in\mathbb{N}}$, converging to a solution of (2.2)

Theorem 3 *Let $d \in \mathbb{N}$. Any sequence $\{e_n\}_{n\in\mathbb{N}} \subset L^1(\mathbb{R}^d)$ of non-negative functions such that*

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} e_n \, dx = I > 0,$$

admits a subsequence, denoted again by $\{e_n\}_{n\in\mathbb{N}}$, for which one of the following phenomena occurs:

- ***Vanishing:*** *For each* $r > 0$*, one has*

$$\lim_{n\to\infty}\left(\sup_{x\in\mathbb{R}^d}\int_{B_r(x)} e_n \,\mathrm{d}x\right) = 0.$$

- ***Dichotomy:*** *There are sequences* $\{x_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^d$*,* $\{M_n\}_{n\in\mathbb{N}}, \{N_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$ *and* $I^* \in (0, I)$ *such that* $M_n, N_n \to \infty$*,* $\frac{M_n}{N_n} \to 0$ *and*

$$\lim_{n\to\infty}\int_{B_{M_n}(x_n)} e_n \,\mathrm{d}x = I^*, \quad \lim_{n\to\infty}\int_{B_{N_n}(x_n)} e_n \,\mathrm{d}x = I^*.$$

- ***Concentration:*** *There exists a sequence* $\{x_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^d$ *with the property that for each* $\varepsilon > 0$*, there exists* $r > 0$ *with*

$$\int_{B_r(x_n)} e_n \,\mathrm{d}x \geq I - \varepsilon, \quad \textit{for all} \;\; n \in \mathbb{N}.$$

Interpreting I as a mass, Theorem 3 says that $\{e_n\}_{n\in\mathbb{N}}$ admits a subsequence for which one of the following occur: The mass spreads out in $\mathbb{R}^n$ (vanishing), it splits into two parts (dichotomy) or the mass is uniformly concentrated in $\mathbb{R}^n$ (concentration).

2.1 Preliminary results

In this subsection we will gather some of the results we need in order to apply Theorem 3.

Proposition 1 *Let* $\phi \in X_{\frac{\alpha}{2}}$*. Then,*

(i) $\mathcal{L}(\phi) = \frac{1}{2}\,\|\phi\|^2_{\frac{\alpha}{2}}$,

(ii) for $\frac{4}{5} \leq \alpha \leq 2$*, one has the anisotropic Sobolev inequality*

$$\|\phi\|^3_{L^3(\mathbb{R}^2)} \lesssim \|\phi\|^{\frac{5\alpha-4}{\alpha+2}}_{L^2(\mathbb{R}^2)} \left\|\mathrm{D}_x^{\frac{\alpha}{2}}\phi\right\|^{\frac{18-5\alpha}{2(\alpha+2)}}_{L^2(\mathbb{R}^2)} \left\|\partial_x^{-1}\partial_y\phi\right\|^{\frac{1}{2}}_{L^2(\mathbb{R}^2)}.$$

In particular $X_{\frac{\alpha}{2}} \subset L^3(\mathbb{R}^2)$ *and* $\|\phi\|_{L^3(\mathbb{R}^2)} \lesssim \|\phi\|_{\frac{\alpha}{2}}$ *for all* $\alpha \geq \frac{4}{5}$*.*

Proof Part (i) is immediate while part (ii) can be found in [26, Lemma 1.1]. □

Corollary 1 *The minimum* I_μ *is positive.*

Proof By Proposition 1 we have that

$$\mu = \mathcal{N}(\phi) \lesssim \|\phi\|^3_{L^3} \lesssim \|\phi\|^3_{\frac{\alpha}{2}} \eqsim \mathcal{L}(\phi)^{\frac{3}{2}}.$$

□

Corollary 1 ensures that the minimizer is not given by the trivial solution.

Lemma 1 *For any* $\alpha > 0$*, the space* $X_{\frac{\alpha}{2}}$ *is compactly embedded in* $L^2_{loc}(\mathbb{R}^2)$.

Proof The proof follows essentially the lines in [10, Lemma 3.3]. We include it here for the sake of completeness.

For $\phi \in X_{\frac{\alpha}{2}}$, let $\varphi = \partial_x^{-1}\phi$. From the definition of $X_{\frac{\alpha}{2}}$ we find that $\partial_x\varphi,\ \partial_y\varphi \in L^2(\mathbb{R}^2)$, that is, $\varphi \in \dot{H}^1(\mathbb{R}^2)$. From Poincare's inequality we have that $\dot{H}^1(\mathbb{R}^2)$ is continuously embedded in $\mathrm{BMO}(\mathbb{R}^2)$. It follows from this that $\varphi \in \mathrm{BMO}(\mathbb{R}^2) \subset L^q_{\mathrm{loc}}(\mathbb{R}^2)$ for all $0 < q < \infty$. Let $\{\phi_n\}_{n=1}^{\infty}$ be a bounded sequence in $X_{\frac{\alpha}{2}}$. We will show that for any $R > 0$ there exists a subsequence $\{\phi_{n_k}\}_{n=1}^{\infty}$, which converges in $L^2(B_R)$, where B_R is the ball of radius R centered at the origin in $\mathbb{R}^2$. Let $\varphi_n = \partial_x^{-1}\phi_n$. Since we are only interested in convergence in $L^2(B_R)$, we may assume that φ_n is supported on B_{2R} by multiplying φ_n with a smooth cutoff function ψ such that $\psi \equiv 1$ in B_R and $\mathrm{supp}(\psi) \subset B_{2R}$. It follows then that ϕ_n is supported on B_{2R} as well.

Since $\{\phi_n\}_{n=1}^{\infty}$ is bounded in $X_{\frac{\alpha}{2}}$ we can extract a subsequence, which we still denote by $\{\phi_n\}_{n=1}^{\infty}$, such that $\phi_n \rightharpoonup \phi$, for some $\phi \in X_{\frac{\alpha}{2}}$. Moreover, by replacing ϕ_n with $\phi_n - \phi$, we may assume that $\phi = 0$. Our aim is then to show that

$$\int_{\mathbb{R}^2} |\phi_n|^2 \,\mathrm{d}(x, y) \to 0, \quad \text{as} \quad n \to \infty.$$

Let $R_1 > 0$. We have

$$\begin{aligned}
\int_{\mathbb{R}^2} |\phi_n|^2 \,\mathrm{d}(x, y) &= \int_{\mathbb{R}^2} |\hat{\phi}_n|^2 \,\mathrm{d}(\xi_1, \xi_2) \\
&= \int_{\{|\xi_1| \le R_1,\ |\xi_2| \le R_1^2\}} |\hat{\phi}_n|^2 \,\mathrm{d}(\xi_1, \xi_2) + \int_{\{|\xi_1| \ge R_1\}} |\hat{\phi}_n|^2 \,\mathrm{d}(\xi_1, \xi_2) \\
&\quad + \int_{\{|\xi_1| \le R_1,\ |\xi_2| \ge R_1^2\}} |\hat{\phi}_n|^2 \,\mathrm{d}(\xi_1, \xi_2).
\end{aligned} \tag{2.4}$$

We proceed to estimate each integral on the right-hand side of (2.4) separately. For the third integral we can write

$$\begin{aligned}
\int_{\{|\xi_1| \le R_1,\ |\xi_2| \ge R_1^2\}} |\hat{\phi}_n|^2 \,\mathrm{d}(\xi_1, \xi_2) &= \int_{\{|\xi_1| \le R_1,\ |\xi_2| \ge R_1^2\}} \frac{\xi_1^2}{\xi_2^2} |\mathcal{F}(\partial_x^{-1}\partial_y\phi_n)|^2 \,\mathrm{d}(\xi_1, \xi_2) \\
&\le \frac{R_1^2}{R_1^4} \left\| \partial_x^{-1}\partial_y\phi_n \right\|^2_{L^2(\mathbb{R}^2)} \\
&= \frac{1}{R_1^2} \left\| \partial_x^{-1}\partial_y\phi_n \right\|^2_{L^2(\mathbb{R}^2)}
\end{aligned}$$

and for the second one

$$\int_{\{|\xi_1| \ge R_1\}} |\hat{\phi}_n|^2 \,\mathrm{d}(\xi_1, \xi_2) = \int_{\{|\xi_1| \ge R_1\}} \frac{1}{|\xi_1|^{\alpha}} |\mathcal{F}(\mathrm{D}_x^{\frac{\alpha}{2}}\phi_n)|^2 \,\mathrm{d}(\xi_1, \xi_2)$$

$$\leq \frac{1}{R_1^\alpha} \left\| \mathrm{D}_x^{\frac{\alpha}{2}} \phi_n \right\|_{L^2(\mathbb{R}^2)}^2 .$$

From these estimates we conclude that, given $\varepsilon > 0$ we can choose R_1 sufficiently large such that

$$\int_{\{|\xi_1| \geq R_1\}} |\hat{\phi}_n|^2 \, \mathrm{d}(\xi_1, \xi_2) + \int_{\{|\xi_1| \leq R_1, \ |\xi_2| \geq R_1^2\}} |\hat{\phi}_n|^2 \, \mathrm{d}(\xi_1, \xi_2) < \varepsilon.$$

In order to deal with the first integral, we first note that since $\phi_n \rightharpoonup 0$ in $X_{\frac{\alpha}{2}}$, we have

$$\hat{\phi}_n(\xi_1, \xi_2) = \int_{B_{2R}} \mathrm{e}^{-\mathrm{i}(x\xi_1 + y\xi_2)} \phi_n(x, y) \, \mathrm{d}(x, y) \to 0 \ \text{ as } \ n \to \infty.$$

Moreover,

$$|\hat{\phi}_n(\xi_1, \xi_2)| \leq \|\phi_n\|_{L^1(B_{2R})} \lesssim \|\phi_n\|_{L^2(B_{2R})} .$$

Since $\{\phi_n\}_{n=1}^\infty$ is bounded in $X_{\frac{\alpha}{2}}$ we can conclude that $\{\hat{\phi}_n\}_{n=1}^\infty$ is bounded in $L^\infty(\mathbb{R}^2)$, so by the dominated convergence theorem

$$\int_{\{|\xi_1| \leq R_1, \ |\xi_2| \leq R_1^2\}} |\hat{\phi}_n|^2 \, \mathrm{d}(\xi_1, \xi_2) \to 0, \ \text{ as } n \to \infty.$$

□

Next we prove that I_μ is subadditive as a function of μ, a property which will be crucial when proving that the dichotomy scenario in Theorem 3 does not occur.

Proposition 2 *The infimum I_μ is strictly increasing and subadditive as a function of μ, that is*

$$I_{\mu_1 + \mu_2} < I_{\mu_1} + I_{\mu_2}, \ \ \textit{for all} \ \ \mu_1, \mu_2 > 0.$$

Proof Let $h \in X_{\frac{\alpha}{2}}$ be such that $\mathcal{N}(h) = 1$ and let $\phi = \mu^{\frac{1}{3}} h$. Then $N(\phi) = \mu$ and $\mathcal{L}(\phi) = \mu^{\frac{2}{3}} \mathcal{L}(h)$, which implies

$$I_\mu = \mu^{\frac{2}{3}} I_1,$$

from which the statement in the proposition directly follows. □

When applying Theorem 3 we will be taking integrals over bounded domains. It is therefore useful to consider the norm $\|\cdot\|_{\frac{\alpha}{2}}$ restricted to a bounded domain $\Omega \subset \mathbb{R}^2$:

$$||\phi||_{\frac{\alpha}{2}, \Omega}^2 = \|\phi\|_{L^2(\Omega)}^2 + \left\| \mathrm{D}_x^{\frac{\alpha}{2}} \phi \right\|_{L^2(\Omega)}^2 + \left\| \partial_x^{-1} \partial_y \phi \right\|_{L^2(\Omega)}^2 .$$

We also make the following definition.

Definition 1 Let Ω be a bounded domain in $\mathbb{R}^2$. For $f \in L^1_{\mathrm{loc}}(\mathbb{R}^2)$, let

$$f_\Omega := f - M_\Omega(f),$$

where $M_\Omega(f) := \frac{1}{|\Omega|}\int_\Omega f \, \mathrm{d}(x, y)$ is the mean of f over Ω.

When proving that the vanishing scenario does not occur we will make use of the following result.

Proposition 3 *Let $\phi \in X_{\frac{\alpha}{2}}$, $\varphi = \partial_x^{-1}\phi$ and let ψ be a smooth cutoff function supported on a bounded domain*[1] $\Omega = \{(x, y) \in \mathbb{R}^2 \mid y \in (a, b), x \in (h_1(y), h_2(y))\}$, *for some $a, b \in \mathbb{R}$ and $h_i \in C([a, b])$, $i = 1, 2$. Define*

$$F_\Omega(\phi) = \partial_x(\psi(\varphi_\Omega)).$$

Then,

$$\|F_\Omega(\phi)\|_{\frac{\alpha}{2}} \lesssim \|\phi\|_{\frac{\alpha}{2},\Omega}.$$

Proof We have that

$$\|F_\Omega(\phi)\|^2_{\frac{\alpha}{2}} = \|\partial_x(\psi\varphi_\Omega)\|^2_{L^2(\mathbb{R}^2)} + \left\|\partial_y(\psi\varphi_\Omega)\right\|^2_{L^2(\mathbb{R}^2)} + \left\|\mathrm{D}_x^{\frac{\alpha}{2}}\partial_x(\psi\varphi_\Omega)\right\|^2_{L^2(\mathbb{R}^2)}.$$

We consider each of these terms separately. The first term can be estimated as

$$\|\partial_x(\psi\varphi_\Omega)\|_{L^2(\mathbb{R}^2)} = \|\psi_x\varphi_\Omega + \psi\phi\|_{L^2(\mathbb{R}^2)} \le \|\psi_x\varphi_\Omega\|_{L^2(\mathbb{R}^2)} + \|\psi\phi\|_{L^2(\mathbb{R}^2)},$$

and $\|\psi\phi\|_{L^2(\mathbb{R}^2)} \lesssim \|\phi\|_{L^2(\Omega)}$, while

$$\begin{aligned}
\|\psi_x\varphi_\Omega\|_{L^2(\mathbb{R}^2)} &\lesssim \|\varphi_\Omega\|_{L^2(\Omega)} \\
&\lesssim \|\varphi_x\|_{L^2(\Omega)} + \left\|\varphi_y\right\|_{L^2(\Omega)} \\
&= \|\phi\|_{L^2(\Omega)} + \left\|\partial_x^{-1}\partial_y\phi\right\|_{L^2(\Omega)},
\end{aligned}$$

where we used Poincar's inequality and the definition $\varphi = \partial_x^{-1}\phi$. Hence,

$$\|\partial_x(\psi\varphi_\Omega)\|^2_{L^2} \lesssim \|\phi\|_{L^2(\Omega)} + \left\|\partial_x^{-1}\partial_y\phi\right\|_{L^2(\Omega)} \tag{2.5}$$

and in the same way we find

$$\left\|\partial_y(\psi\varphi_\Omega)\right\|^2_{L^2} \lesssim \|\phi\|_{L^2(\Omega)} + \left\|\partial_x^{-1}\partial_y\phi\right\|_{L^2(\Omega)}. \tag{2.6}$$

[1] The proposition can be generalized to domains, which are given by disjoint unions of type Ω.

Recall that $\Omega = \{(x, y) \in \mathbb{R}^2 \mid y \in (a, b), x \in (h_1(y), h_2(y))\}$ and set $\Omega_y := (h_1(y), h_2(y))$. By the Leibniz rule for fractional derivatives (see e.g. [17, Theorem 7.6.1]), we can estimate

$$\begin{aligned}\left\|\mathrm{D}_x^{\frac{\alpha}{2}}\partial_x(\psi\varphi_\Omega)\right\|_{L^2(\mathbb{R}^2)}^2 &= \int_{\mathbb{R}}\left\|\mathrm{D}_x^{\frac{\alpha}{2}}\partial_x(\psi\varphi_\Omega)(\cdot,y)\right\|_{L^2(\Omega_y)}^2 \mathrm{d}y\\ &\lesssim \int_a^b \left\|\mathrm{D}_x^{\frac{\alpha}{2}}\partial_x\psi(\cdot,y)\right\|_{L^\infty(\Omega_y)}^2 \|\varphi_\Omega(\cdot,y)\|_{L^2(\Omega_y)}^2\\ &\quad + \|\psi(\cdot,y)\|_{L^\infty(\Omega_y)}^2\left\|\mathrm{D}_x^{\frac{\alpha}{2}}\partial_x\varphi_\Omega(\cdot,y)\right\|_{L^2(\Omega_y)}^2 \mathrm{d}y.\end{aligned}$$

Using that ψ is a smooth function, we conclude by Poincaré's inequality that

$$\begin{aligned}\left\|\mathrm{D}_x^{\frac{\alpha}{2}}\partial_x(\psi\varphi_\Omega)\right\|_{L^2(\mathbb{R}^2)}^2 &\lesssim \|\varphi_\Omega\|_{L^2(\Omega)}^2 + \left\|\mathrm{D}_x^{\frac{\alpha}{2}}\partial_x\varphi_\Omega\right\|_{L^2(\Omega)}^2\\ &\lesssim \|\varphi_x\|_{L^2(\Omega)} + \left\|\varphi_y\right\|_{L^2(\Omega)} + \left\|\mathrm{D}_x^{\frac{\alpha}{2}}\partial_x\varphi\right\|_{L^2(\Omega)}^2\\ &= \|\phi\|_{L^2(\Omega)} + \left\|\partial_x^{-1}\partial_y\phi\right\|_{L^2(\Omega)} + \left\|\mathrm{D}_x^{\frac{\alpha}{2}}\phi\right\|_{L^2(\Omega)}^2.\end{aligned} \tag{2.7}$$

Gathering (2.5), (2.6), and (2.7), we have shown that

$$\|F_\Omega(\phi)\|_{\frac{\alpha}{2}} \lesssim \|\phi\|_{\frac{\alpha}{2},\Omega}.$$

□

Eventually, when excluding the dichotomy scenario we will make use of the following lemma, which provides a Poincaré-like inequality.

Lemma 2 *([9], Lemma 3.1) Let $2 \le p < \infty$ and $R > 0$. Then there exists a positive constant C such that for all $f \in L^1_{loc}(\mathbb{R}^2)$ one has that*

$$\|f_{A_{2R,R}}\|_{L^p(A_{2R,R})} \le C R^{\frac{2}{p}} \|\nabla f\|_{L^2(A_{2R,R})},$$

where $A_{2R,R} \subset \mathbb{R}^2$ denotes the annulus centered at the origin of radii $2R > R$.

2.2 Existence of minimizers

Let $\{\phi_n\}_{n\in\mathbb{N}} \subset X_{\frac{\alpha}{2}}$ be a minimizing sequence for the constrained minimization problem (2.2), that is, $\mathcal{N}(\phi_n) = \mu$ and $\lim_{n\to\infty} \mathcal{L}(\phi_n) = I_\mu$. We will apply Theorem 3 to the sequence

$$e_n = \frac{1}{2}\left(\phi_n^2 + (\mathrm{D}_x^{\frac{\alpha}{2}}\phi_n)^2 + (\partial_x^{-1}\partial_y\phi_n)^2\right).$$

Recall that

$$\int_{\mathbb{R}^2} e_n \,\mathrm{d}(x, y) = \mathcal{L}(\phi_n).$$

We will show in Proposition 4 and Proposition 5 that the vanishing and dichotomy scenarios in Theorem 3 does not occur and then use the concentration scenario to construct a convergent subsequence of $\{\phi_n\}_{n\in\mathbb{N}}$, converging to a solution ϕ of (2.2).

Proposition 4 *(Excluding "vanishing") No subsequence of* $\{e_n\}_{n\in\mathbb{N}}$ *has the* vanishing *property in Theorem 3.*

Proof Assume for a contradiction that vanishing does occur, that is

$$\lim_{n\to\infty}\left(\sup_{(x,y)\in\mathbb{R}^2}\int_{B_r(x,y)} e_n \,\mathrm{d}(x, y)\right) = 0$$

for each $r > 0$. Let us cover $\mathbb{R}^2$ with balls $B_{1,j}$, $j \in \mathbb{N}$, of radius 1 such that each point in $\mathbb{R}^2$ is contained in at most three balls. Let $\{\psi_j\}_{n\in\mathbb{N}}$ be a smooth partition of unity such that $\operatorname{supp}(\psi_j) \subset B_{1,j}$. Using Proposition 1 (ii) and Proposition 3 we find

$$\begin{aligned}
|\mathcal{N}(\phi_n)| &\lesssim \|\phi_n\|^3_{L^3(\mathbb{R}^2)} \\
&\lesssim \sum_{j\in\mathbb{N}} \left\|F_{B_{1,j}}(\phi_n)\right\|^3_{L^3(\mathbb{R}^2)} \\
&\le \sup_{j\in\mathbb{N}} \left\|F_{B_{1,j}}(\phi_n)\right\|_{L^3(\mathbb{R}^2)} \sum_{j\in\mathbb{N}} \left\|F_{B_{1,j}}(\phi_n)\right\|^2_{L^3(\mathbb{R}^2)} \\
&\lesssim \sup_{j\in\mathbb{N}} \left\|F_{B_{1,j}}(\phi_n)\right\|_{\frac{\alpha}{2}} \sum_{j\in\mathbb{N}} \left\|F_{B_{1,j}}(\phi_n)\right\|^2_{\frac{\alpha}{2}} \\
&\lesssim \sup_{j\in\mathbb{N}} \|\phi_n\|_{\frac{\alpha}{2},B_{1,j}} \sum_{j\in\mathbb{N}} \|\phi_n\|^2_{\frac{\alpha}{2},B_{1,j}} \\
&\lesssim \sup_{j\in\mathbb{N}} \left(\int_{B_{1,j}} e_n \,\mathrm{d}(x, y)\right)^{\frac{1}{2}} \|\phi_n\|^2_{\frac{\alpha}{2}}\,.
\end{aligned}$$

By letting $n \to \infty$ we get $\mathcal{N}(\phi_n) \to 0$, which contradicts the fact that $\mathcal{N}(\phi_n) = \mu > 0$. □

Proposition 5 *(Excluding "dichotomy") No subsequence of* $\{e_n\}_{n\in\mathbb{N}}$ *has the* dichotomy *property in Theorem 3.*

Proof Throughout the proof we will use B_R to denote the ball in $\mathbb{R}^2$ centered at the origin of radius $R > 0$ and A_{R_1,R_2} to denote the annulus centered at the origin of radii $R_1 > R_2 > 0$.

Assume for a contradiction that the dichotomy scenario in Theorem 3 occurs, that is there exist sequences $\{(x_n, y_n)\}_{n\in\mathbb{N}} \subset \mathbb{R}^2$, $\{M_n\}_{n\in\mathbb{N}}$, $\{N_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$ and $I^* \in (0, I_\mu)$ with $M_n, N_n, \frac{N_n}{M_n} \to \infty$ for $n \to \infty$ and

$$\lim_{n\to\infty} \int_{B_{M_n}(x_n,y_n)} e_n \,\mathrm{d}x = I^*, \quad \lim_{n\to\infty} \int_{B_{N_n}(x_n,y_n)} e_n \,\mathrm{d}x = I^*. \tag{2.8}$$

We will show that this leads to a contradiction, by proving that provided (2.8) holds, there exists two sequences $\{\omega_n^{(1)}\}_{n\in\mathbb{N}}$, $\{\omega_n^{(2)}\}_{n\in\mathbb{N}}$, which have in the limit $n \to \infty$ disjoint support and

(i) $\mathcal{N}(\omega_n^{(1)}) + \mathcal{N}(\omega_n^{(2)}) - \mathcal{N}(\omega_n) \to 0$,
(ii) $\mathcal{L}(\omega_n^{(1)}) \to I^*$ and $\mathcal{L}(\omega_n^{(2)}) \to (I_\mu - I^*)$,

where $\omega_n = \phi_n(\cdot + (x_n, y_n))$ is the shift of ϕ_n by (x_n, y_n). We shift the function ϕ_n for reasons of convenience in order to work with balls and annuli centered at the origin instead of at (x_n, y_n). Notice that if (i) and (ii) hold we obtain a contradiction due to the subadditivity of the I_μ stated in Proposition 2: Set

$$\mu_{1,n} := \mathcal{N}(\omega_n^{(1)}) \qquad \text{and} \qquad \mu_{2,n} := \mathcal{N}(\omega_n^{(2)}),$$

and $\mu_i := \lim_{n\to\infty} \mu_{i,n}$ for $i = 1, 2$. Then (i) implies that $\mu_1 + \mu_2 = \mu$, since $\mathcal{N}(\omega_n) = \mu$ for all $n \in \mathbb{N}$. First we show that $\mu_1 \neq 0$. If $\mu_1 = 0$, then $\mu_2 = \mu$. By setting

$$\tilde{\omega}_n^{(2)} := \left(\frac{\mu}{\mu_{2,n}}\right)^{\frac{1}{3}} \omega_n^{(2)}$$

we find $\mathcal{N}(\tilde{\omega}_n^{(2)}) = \mu$ for all $n \in \mathbb{N}$ and

$$\left|\mathcal{L}(\tilde{\omega}_n^{(2)}) - \mathcal{L}(\omega_n^{(2)})\right| \to 0 \qquad \text{for} \quad n \to \infty,$$

since $\lim_{n\to\infty} \frac{\mu}{\mu_{2,n}} = 1$. But then by using (ii) we obtain

$$I_\mu \le \mathcal{L}(\tilde{\omega}_n^{(2)}) \to I_\mu - I^* < I_\mu \qquad \text{for} \quad n \to \infty,$$

which is a contradiction. Hence, $\mu_1 \neq 0$ and similarly we find $\mu_2 \neq 0$. Thus, $|\mu_i| > 0$ for $i = 1, 2$ and we can define the rescaled functions

$$\bar{\omega}_n^{(i)} := \left(\frac{|\mu_i|}{\mu_{i,n}}\right)^{\frac{1}{3}} \omega_n^{(i)} \qquad \text{for} \quad i = 1, 2,$$

which satisfy $\mathcal{N}(\bar{\omega}_n^{(i)}) = |\mu_i|$ for all $n \in \mathbb{N}$ and

$$\lim_{n\to\infty} \mathcal{L}(\bar{\omega}_n^{(1)}) = I^*, \qquad \lim_{n\to\infty} \mathcal{L}(\bar{\omega}_n^{(1)}) = I_\mu - I^*,$$

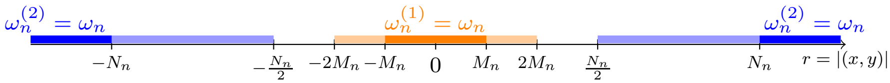

Fig. 1 For n large enough the supports of $\omega_n^{(1)}$ and $\omega_n^{(2)}$, given by B_{2M_n} and $\mathbb{R}^2 \setminus B_{\frac{N_n}{2}}$, are disjoint. On B_{M_n} and $\mathbb{R}^2 \setminus B_{N_n}$ the functions $\omega_n^{(1)}$ and $\omega_n^{(2)}$ coincide with ω_n, respectively

by (ii) together with $\lim_{n\to\infty} \left|\frac{|\mu_i|}{\mu_{i,n}}\right| = 1$. Combining this with the subadditivity of I_μ for $\mu > 0$, which is stated in Proposition 2, we find the contradiction

$$I_\mu \le I_{|\mu_1|+|\mu_2|} < I_{|\mu_1|} + I_{|\mu_2|} \le \lim_{n\to\infty} \left(\mathcal{L}(\bar{\omega}_n^{(1)}) + \mathcal{L}(\bar{\omega}_n^{(2)})\right) = I_\mu.$$

We are left to show that there exists two sequences $\{\omega_n^{(1)}\}_{n\in\mathbb{N}}$, $\{\omega_n^{(2)}\}_{n\in\mathbb{N}}$, which have in the limit $n \to \infty$ disjoint support and satisfy (i), (ii). To this end, let $\varphi_n = \partial_x^{-1}\phi_n$ and let $\chi : \mathbb{R}^2 \to [0, 1]$ be a smooth cutoff function such that $\chi(x, y) = 1$ for $|(x, y)| \le 1$ and $\chi(x, y) = 0$ for $|(x, y)| \ge 2$. Next let $\sigma_n := \varphi_n(\cdot + (x_n, y_n))$ and

$$\sigma_n^{(1)} := \chi_{1n}\sigma_{n,A_{2M_n,M_n}}, \quad \sigma_n^{(2)} := \chi_{2n}\sigma_{n,A_{N_n,N_n/2}},$$

where

$$\chi_{1n}(x, y) := \chi\left(\frac{1}{M_n}(x, y)\right), \quad \chi_{2n} := 1 - \chi\left(\frac{2}{N_n}(x, y)\right).$$

Eventually, we define

$$\omega_n := \partial_x\sigma_n, \quad \omega_n^{(i)} := \partial_x\sigma_n^{(i)}, \ i = 1, 2.$$

We remark that by definition $\omega_n = \phi_n(\cdot + (x_n, y_n))$. Furthermore,

$$\operatorname{supp}(\omega_n^{(1)}) \subset B_{2M_n} \quad \text{and} \quad \operatorname{supp}(\omega_n^{(2)}) \subset \mathbb{R}^2 \setminus B_{\frac{N_n}{2}}.$$

See Figure 1 for an illustration of the supports for $\omega_n^{(i)}$, $i = 1, 2$.

Roughly speaking the dichotomy assumption implies that the mass of e_n, which is given by $\mathcal{L}(\phi_n) = \frac{1}{2}\|\phi_n\|^2_{X_{\frac{\alpha}{2}}}$ splits into two disjoint regions. To be more precise, (2.8) yields

$$\begin{aligned}\|\omega_n\|^2_{\frac{\alpha}{2},A_{N_n,M_n}} &= \|\omega_n\|^2_{\frac{\alpha}{2},B_{N_n}} - \|\omega_n\|^2_{\frac{\alpha}{2},B_{M_n}} \\ &= 2\left(\int_{(x_n,y_n)+B_{N_n}} e_n \,\mathrm{d}(x, y) - \int_{(x_n,y_n)+B_{M_n}} e_n \,\mathrm{d}(x, y)\right) \to 0,\end{aligned} \tag{2.9}$$

as $n \to \infty$. Using this result together with Proposition 1 (ii) and Proposition 3 we also find that

$$\|\omega_n\|_{L^3(A_{N_n,M_n})} \to 0 \quad \text{for} \quad n \to \infty. \tag{2.10}$$

In what follows we will prove that the statements (i) and (ii) hold true.

(i) Consider

$$\begin{aligned} &\left|\mathcal{N}(\omega_n^{(1)}) + \mathcal{N}(\omega_n^{(2)}) - \mathcal{N}(\omega_n)\right| \\ &= \left|\int_{\mathbb{R}^2} (\omega_n^{(1)})^3 \,\mathrm{d}(x,y) + \int_{\mathbb{R}^2} (\omega_n^{(2)})^3 \,\mathrm{d}(x,y) - \int_{\mathbb{R}^2} \omega_n^3 \,\mathrm{d}(x,y)\right| \\ &= \left|\int_{A_{2M_n,M_n}} (\omega_n^{(1)})^3 \mathrm{d}(x,y) \quad + \int_{A_{N_n,\frac{N_n}{2}}} (\omega_n^{(2)})^3 \,\mathrm{d}(x,y)\right. \\ &\quad \left. - \int_{A_{N_n,M_n}} \omega_n^3 \,\mathrm{d}(x,y)\right|, \end{aligned} \tag{2.11}$$

where we used that $\omega_n^{(1)} = \omega_n$ on B_{M_n} and $\omega_n^{(2)} = \omega_n$ on $\mathbb{R}^2 \setminus B_{N_n}$. The term $\int_{A_{N_n,M_n}} \omega_n^3 \,\mathrm{d}(x,y)$ tends to zero in view of (2.10) and

$$\begin{aligned} \left\|w_n^{(1)}\right\|_{L^3(A_{2M_n,M_n})} &= \left\|\partial_x \sigma_n^{(1)}\right\|_{L^3(A_{2M_n,M_n})} \\ &\le \frac{1}{M_n} \left\|\partial_x \chi_{1n} \sigma_{n,A_{2M_n,M_n}}\right\|_{L^3(A_{2M_n,M_n})} \\ &\quad + \left\|\chi_{1n} \partial_x \sigma_{n,A_{2M_n,M_n}}\right\|_{L^3(A_{2M_n,M_n})} \\ &= \frac{1}{M_n} \left\|\partial_x \chi_{1n} \sigma_{n,A_{2M_n,M_n}}\right\|_{L^3(A_{2M_n,M_n})} \\ &\quad + \|\chi_{1n} \omega_n\|_{L^3(A_{2M_n,M_n})}, \end{aligned}$$

where we used that $\partial_x \sigma_{n,A_{2M_n,M_n}} = \partial_x \sigma_n = \omega_n$. Using Lemma 2, the smoothness of $\chi_{1,n}$, and (2.9), the first term on the right-hand side above can be estimated by

$$\begin{aligned} &\frac{1}{M_n} \left\|\partial_x \chi_{1n} \sigma_{n,A_{2M_n,M_n}}\right\|_{L^3(A_{2M_n,M_n})} \lesssim \frac{1}{M_n} \left\|\sigma_{n,A_{2M_n,M_n}}\right\|_{L^3(A_{2M_n,M_n})} \\ &\lesssim M_n^{-\frac{2}{3}} \left\|\nabla \sigma_{n,A_{2M_n,M_n}}\right\|_{L^2(A_{2M_n,M_n})} \\ &\le M_n^{-\frac{2}{3}} \left(\left\|\partial_x \sigma_{n,A_{2M_n,M_n}}\right\|_{L^2(A_{2M_n,M_n})} + \left\|\partial_y \sigma_{n,A_{2M_n,M_n}}\right\|_{L^2(A_{2M_n,M_n})}\right) \\ &= M_n^{-\frac{2}{3}} \left(\|\omega_n\|_{L^2(A_{2M_n,M_n})} + \left\|\partial_y \partial_x^{-1} \omega_n\right\|_{L^2(A_{2M_n,M_n})}\right) \\ &\le M_n^{-\frac{2}{3}} \|\omega_n\|_{\frac{\alpha}{2},A_{N_n,M_n}} \to 0 \end{aligned} \tag{2.12}$$

as $n \to \infty$. The second term tends to zero as $n \to \infty$ due to (2.10) and the boundedness of $\chi_{1,n}$. We conclude

$$\int_{\mathbb{R}^2} (\omega_n^{(1)})^3 \,\mathrm{d}(x,y) \to 0 \quad \text{and} \quad \int_{\mathbb{R}^2} (\omega_n^{(2)})^3 \,\mathrm{d}(x,y) \to 0$$

as $n \to \infty$, where the second assertion can be shown in the same way. Together with (2.10), equation (2.11) finishes the proof of statement (i).

(ii) We proceed to investigate the limit

$$\begin{aligned}&\lim_{n\to\infty} \mathcal{L}(\omega_n^{(1)})\\ &= \frac{1}{2}\lim_{n\to\infty}\left(\left\|\omega_n^{(1)}\right\|^2_{L^2(\mathbb{R}^2)} + \left\|\mathrm{D}_x^{\frac{\alpha}{2}}\omega_n^{(1)}\right\|^2_{L^2(\mathbb{R}^2)} + \left\|\partial_x^{-1}\partial_y\omega_n^{(1)}\right\|^2_{L^2(\mathbb{R}^2)}\right)\end{aligned}$$

and show that $\lim_{n\to\infty} \mathcal{L}(\omega_n^{(1)}) = I^*$. First consider

$$\begin{aligned}\left\|\omega_n^{(1)}\right\|^2_{L^2(\mathbb{R}^2)} &= \left\|\frac{1}{M_n}\partial_x\chi_{1n}\sigma_{n,B_{M_n}} + \chi_{1n}\omega_n\right\|^2_{L^2(\mathbb{R}^2)}\\ &= \frac{1}{M_n^2}\left\|\partial_x\chi_{1n}\sigma_{n,A_{2M_n,M_n}}\right\|^2_{L^2(\mathbb{R}^2)} + \frac{2}{M_n}\langle\partial_x\chi_{1n}\sigma_{n,A_{2M_n,M_n}},\chi_{1n}\omega_n\rangle_{L^2(\mathbb{R}^2)}\\ &+ \|\chi_{1n}\omega_n\|^2_{L^2(\mathbb{R}^2)}.\end{aligned} \tag{2.13}$$

Since $\partial_x\chi_{1,n}$ has support in A_{2M_n,M_n} a similar argument as in (2.12) shows

$$\frac{1}{M_n}\left\|\partial_x\chi_{1n}\sigma_{n,A_{2M_n,M_n}}\right\|_{L^2(\mathbb{R}^2)} \lesssim \|\omega_n\|_{\frac{\alpha}{2},A_{2M_n,M_n}} \to 0, \text{ as } n\to\infty, \tag{2.14}$$

by using Lemma 2 and (2.9). Hence, we find that both the first and second term on the right-hand side of (2.13) tend to zero as $n \to \infty$. For the third term on the right-hand side of (2.13) we have

$$\|\chi_{1n}\omega_n\|^2_{L^2(\mathbb{R}^2)} = \|\omega_n\|^2_{L^2(B_{M_n})} + \|\chi_{1n}\omega_n\|^2_{L^2(A_{2M_n,M_n})},$$

where we used that $\mathrm{supp}(\chi_{1,n}) \subset B_{2M_n}$ and $\chi_{1,n} = 1$ on B_{M_n}. Due to (2.9) we find

$$\|\chi_{1n}\omega_n\|_{L^2(A_{2M_n,M_n})} \lesssim \|\omega_n\|_{\frac{\alpha}{2},A_{2M_n,M_n}} \to 0.$$

We conclude

$$\lim_{n\to\infty}\left|\left\|\omega_n^{(1)}\right\|^2_{L^2(\mathbb{R}^2)} - \|\omega_n\|^2_{L^2(B_{M_n})}\right| = 0. \tag{2.15}$$

In the same way we can show

$$\lim_{n\to\infty}\left|\left\|\partial_x^{-1}\partial_y\omega_n^{(1)}\right\|_{L^2(\mathbb{R}^2)}-\left\|\partial_x^{-1}\partial_y\omega_n\right\|^2_{L^2(B_{M_n})}\right|=0, \tag{2.16}$$

so that we are only left to study

$$\begin{aligned}\left\|\mathrm{D}_x^{\frac{\alpha}{2}}\omega_n^{(1)}\right\|^2_{L^2(\mathbb{R}^2)}&=\left\|\mathrm{D}_x^{\frac{\alpha}{2}}\left(\frac{1}{M_n}\partial_x\chi_{1n}\sigma_{n,A_{2M_n,M_n}}+\chi_{1n}\omega_n\right)\right\|^2_{L^2(\mathbb{R}^2)}\\&=\frac{1}{M_n^2}\left\|\mathrm{D}_x^{\frac{\alpha}{2}}(\partial_x\chi_{1n}\sigma_{n,A_{2M_n,M_n}})\right\|^2_{L^2(\mathbb{R}^2)}\\&\quad+\frac{2}{M_n}\langle\mathrm{D}_x^{\frac{\alpha}{2}}(\partial_x\chi_{1n}\sigma_{n,A_{2M_n,M_n}}),\mathrm{D}_x^{\frac{\alpha}{2}}(\chi_{1n}\omega_n)\rangle_{L^2(\mathbb{R}^2)}\\&\quad+\left\|\mathrm{D}_x^{\frac{\alpha}{2}}(\chi_{1n}\omega_n)\right\|^2_{L^2(\mathbb{R}^2)}.\end{aligned}\tag{2.17}$$

We show first

$$\|\mathrm{D}_x^{\frac{\alpha}{2}}\left(\partial_x\chi_{1,n}\sigma_{n,A_{2M_n,M_n}}\right)\|_{L^2(\mathbb{R}^2)}\lesssim M_n\|\omega_n\|_{X_{\frac{\alpha}{2}},A_{2M_n,M_n}}, \tag{2.18}$$

which implies by (2.9), the smoothness of $\chi_{1,n}$ and the boundedness of ω_n in $X_{\frac{\alpha}{2}}$ that the first two terms on the right-hand side of (2.17) tend to zero as $n\to\infty$. As in the proof of Proposition 3, an application of Leibniz' rule for fractional derivatives yields

$$\begin{aligned}&\|\mathrm{D}_x^{\frac{\alpha}{2}}\left(\partial_x\chi_{1,n}\sigma_{n,A_{2M_n,M_n}}\right)\|_{L^2(\mathbb{R}^2)}\\&\lesssim\|\sigma_{n,A_{2M_n,M_n}}\|_{L^2(A_{2M_n,M_n})}+\|\mathrm{D}_x^{\frac{\alpha}{2}}\sigma_{n,A_{2M_n,M_n}}\|_{L^2(A_{2M_n,M_n})}\\&\leq 2\|\sigma_{n,A_{2M_n,M_n}}\|_{L^2(A_{2M_n,M_n})}+\|\mathrm{D}_x^{\frac{\alpha}{2}}\omega_n\|_{L^2(A_{2M_n,M_n})},\end{aligned}$$

where we used interpolation and $\partial_x\sigma_{n,A_{2M_n,M_n}}=\omega_n$ in the last inequality. Using Lemma 2, the first term on the right-hand side above can by estimated by $M_n\|\omega_n\|_{X_{\frac{\alpha}{2}},A_{2M_n,M_n}}$ in the same spirit as in (2.12), while the second term is bounded by $\|\omega_n\|_{X_{\frac{\alpha}{2}},A_{2M_n,M_n}}$. Hence, (2.18) holds true and the first two terms in (2.17) tend to zero as $n\to\infty$.

It remains to investigate the third term in (2.14), given by

$$\|\mathrm{D}_x^{\frac{\alpha}{2}}\left(\chi_{1,n}\omega_n\right)\|^2_{L^2(\mathbb{R}^2)}=\|\mathrm{D}_x^{\frac{\alpha}{2}}\omega_n\|^2_{L^2(B_{M_n})}+\|\mathrm{D}_x^{\frac{\alpha}{2}}\left(\chi_{1,n}\omega_n\right)\|^2_{L^2(A_{2M_n,M_n})}.$$

Again, by applying Leibniz' rule for fractional derivatives and using that $\chi_{1,n}$ is smooth, we find that

$$\begin{aligned}\|\mathrm{D}_x^{\frac{\alpha}{2}}\left(\chi_{1,n}\omega_n\right)\|^2_{L^2(A_{2M_n,M_n})}&\lesssim\|\omega_n\|^2_{L^2(A_{2M_n,M_n})}+\|\mathrm{D}_x^{\frac{\alpha}{2}}\omega_n\|^2_{L^2(A_{2M_n,M_n})}\\&\leq\|\omega\|_{X_{\frac{\alpha}{2}},A_{2M_n,M_n}}\to 0,\end{aligned}$$

by (2.9). This allows us to conclude

$$\lim_{n\to\infty}\left|\left\|\mathrm{D}_x^{\frac{\alpha}{2}}\omega_n^{(1)}\right\| - \left\|\mathrm{D}_x^{\frac{\alpha}{2}}\omega_n\right\|_{L^2(B_{M_n})}\right| = 0. \tag{2.19}$$

Gathering (2.15), (2.16), and (2.19) we have shown

$$\mathcal{L}(\omega_n^{(1)}) \to I^* \quad \text{for} \quad n \to \infty.$$

In the same way we can obtain

$$\mathcal{L}(\omega_n^{(2)}) \to I_\mu - I^* \quad \text{for} \quad n \to \infty,$$

which proves statement (ii).

□

By Proposition 4 and Proposition 5 the scenarios of "vanishing" and "dichotomy" in Theorem 3 are ruled out and the only possibility left is the concentration scenario. Hence, there exists $\{(x_n, y_n)\}_{n\in\mathbb{N}} \subset \mathbb{R}^2$ such that for each $\varepsilon > 0$ there exists $r > 0$ with

$$\int_{B_r(x_n,y_n)} e_n \,\mathrm{d}(x, y) \geq I_\mu - \varepsilon, \text{ for all } n \in \mathbb{N}.$$

This implies that, for r sufficiently large,

$$\int_{\mathbb{R}^2\setminus B_r(x_n,y_n)} e_n < \varepsilon. \tag{2.20}$$

By taking a subsequence we may assume that (2.20) holds for all $n \in \mathbb{N}$. This implies in particular that

$$\|\phi_n(\cdot - (x_n, y_n))\|_{L^2(|(x,y)|>r)} < \varepsilon. \tag{2.21}$$

Since $\{\phi_n\}_{n\in\mathbb{N}}$ is a bounded sequence in $X_{\frac{\alpha}{2}}$ we may assume $\phi_n \rightharpoonup \phi \in X_{\frac{\alpha}{2}}$. From Proposition 1 we know that $X_{\frac{\alpha}{2}}$ is compactly embedded in $L^2_{\text{loc}}(\mathbb{R}^2)$, and therefore $\phi_n \to \phi$ strongly in $L^2_{\text{loc}}(\mathbb{R}^2)$. By combining this with (2.21) we can use Cantors diagonal extraction process to extract a subsequence, still denoted by ϕ_n, converging strongly in $L^2(\mathbb{R}^2)$. Proposition 1 (ii) implies that for $\alpha > \frac{4}{5}$, $\phi_n \to \phi$ in $L^3(\mathbb{R}^2)$ as well, which yields that $\mathcal{N}(\phi) = \mu$. Finally, since $\mathcal{L}(\phi) = \frac{1}{2}\|\phi\|_{\frac{\alpha}{2}}$ and the norm is weakly lower semicontinuous, we find

$$\mathcal{L}(\phi) \leq \liminf \mathcal{L}(\phi_n) = I_\mu.$$

Hence, ϕ is a solution of the minimization problem (2.2).

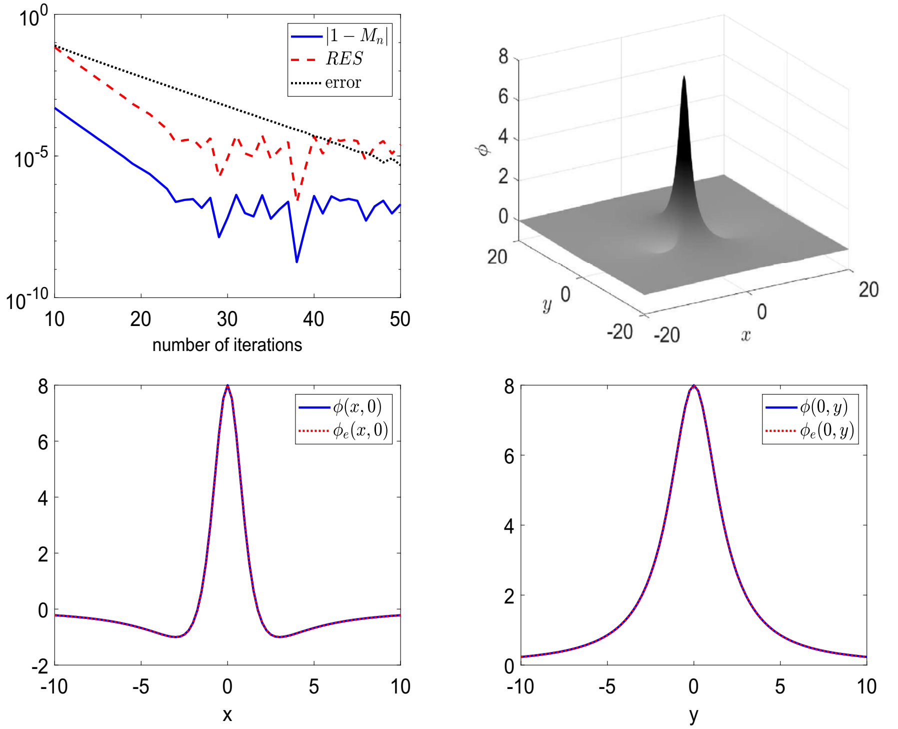

Fig. 2 The variation of the iteration, stabilization factor, and the residual errors with the number of iterations in the semi-log scale (top left), numerically generated lump solution of KP-I equation (top right), x-cross section $\phi(x, 0)$ (bottom left) and y-cross section $\phi(0, y)$ (bottom right) of both numerical and analytical solutions

Next, we investigate the lump solutions numerically. For the implementation of the iteration scheme (1.7), we consider the space interval as $[-1024, 1024] \times [-1024, 1024]$ and we set $c = 1$ in all the experiments. We use $\phi_0(x, y) = \exp(-x^2 - y^2)$ as the initial guess for the iteration. To test the efficiency of the numerical scheme we first consider the KP-I case (i.e., $\alpha = 2$) where the exact analytical lump solution is given in (1.3).

In Figure 2, we represent the numerically generated lump solution of the KP-I equation and the cross sections $\phi(x, 0)$ and $\phi(0, y)$ of both numerical and analytical solutions. Choosing the number of grid points as $N_x = N_y = 2^{13}$ for both x and y coordinates we see that the L^∞-norm of the difference of numerical and exact solutions is approximately of order 10^{-5} after 50 iterations. In Figure 2 we also present the variation of three different errors with the number of iterations in a semi-log scale. The two-dimensional geometry of the solutions and the periodic setting in both directions for implementing the numerical scheme cause a slow convergence rate. In Table 1, we present errors for increasing values of N_x and N_y. We observe that the errors get smaller for larger values of N_x and N_y.

Table 1 Numerical errors for several values of N_x and N_y when $\alpha = 2$ after 50 iterations

$N_x = N_y$	$\|1 - M_n\|$	RES	error
2^{11}	2.072E-7	3.074E-5	1.081E-4
2^{12}	1.708E-7	2.178E-5	4.904E-6
2^{13}	1.072E-8	1.372E-6	6.303E-6

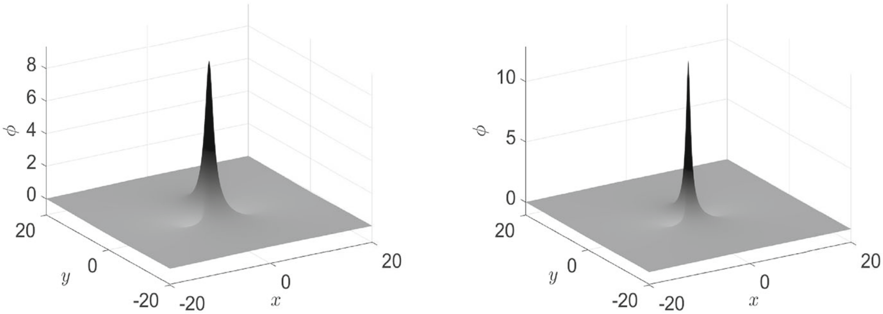

Fig. 3 Lump solutions for $\alpha = 1.7$ (left panel) and $\alpha = 1.35$ (right panel)

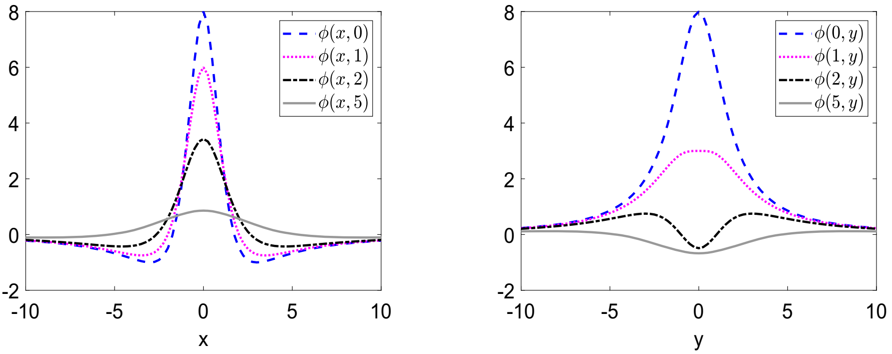

Fig. 4 Several x and $y-$cross sections of the numerically generated lump solution for $\alpha = 2$

In the next experiment, we consider some examples for the fractional case. Figure 3 depicts the profiles of the numerical solutions for $\alpha = 1.7$ and $\alpha = 1.35$, respectively. It can be seen from the numerical results that the lump solutions become more peaked for smaller values of α. Therefore, to ensure the required numerical accuracy we need to increase the number of grid points to 2^{14} for both x and y directions when $\alpha = 1.35$. In this case the Fourier coefficients go down to 10^{-5}. To obtain the same numerical accuracy for smaller values of α, we need to increase the number of Fourier modes even more, which is not accessible due to the limits of computation.

We also observe the cross-sectional symmetry of the lump solutions of the fKP-I equation numerically. We present several x and y-cross sections of the solutions for various α. We consider the cases $\alpha = 2$, $\alpha = 1.7$ and $\alpha = 1.35$ in Figure 4, Figure 5, and Figure 6, respectively. The numerical results indicate symmetry in both x and y directions.

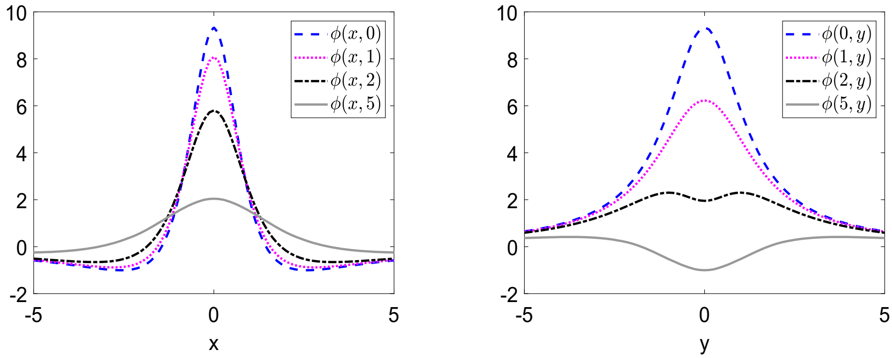

Fig. 5 Several x and $y-$cross sections of the numerically generated lump solution for $\alpha = 1.7$

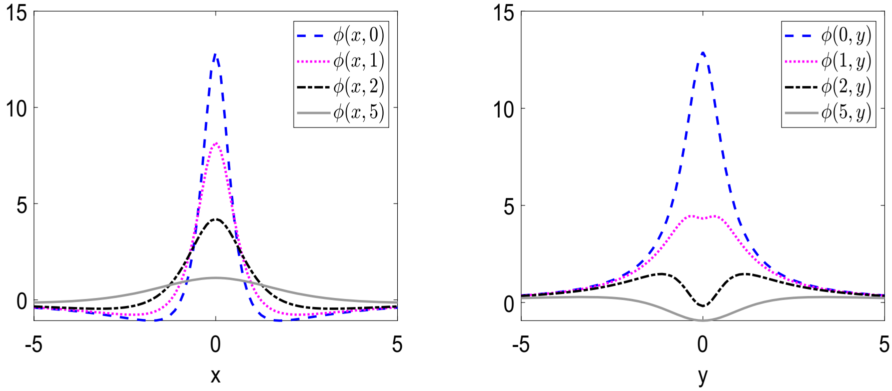

Fig. 6 Several x and $y-$cross sections of the numerically generated lump solution for $\alpha = 1.35$

3 Decay of lump solutions

Throughout this section, unless specifically stated otherwise, we assume that $\alpha > \frac{4}{5}$. The existence of lump solutions $u(t, x, y) = \phi(x-ct, y)$ for the fKP-I equation, where $\phi \in X_{\frac{\alpha}{2}}$, was proved in the previous section. The function ϕ satisfies the (rescaled) traveling wave fKP-I equation

$$-\phi_{xx} - \phi_{yy} - \mathrm{D}_x^{\alpha}\phi_{xx} + \frac{1}{2}(\phi^2)_{xx} = 0, \tag{3.1}$$

which can be written in convolution form as

$$\phi = \frac{1}{2}K_\alpha * \phi^2, \qquad \hat{K}_\alpha(\xi_1, \xi_2) = m_\alpha(\xi_1, \xi_2), \tag{3.2}$$

where the symbol m_α is given by

$$m_\alpha(\xi_1, \xi_2) = \frac{\xi_1^2}{|\xi|^2 + |\xi_1|^{\alpha+2}}.$$

Let us recall from Remark 1 that any nontrivial, continuous solution ϕ of (3.2) decays *at most* quadratically. In this section, we show that any nontrivial solution $\phi \in X_{\frac{\alpha}{2}}$ of (3.2) decays indeed *exactly* quadratically, that is we prove Theorem 2.

The idea is to study the kernel function K_α and to show that it has quadratic decay at infinity (independent of α). Then the decay properties of K_α are used to show that also ϕ decays quadratically at infinity.

In the sequel we denote by $\varrho : \mathbb{R}^2 \to \mathbb{R}$ the function

$$\varrho(x, y) = |(x, y)| = (x^2 + y^2)^{\frac{1}{2}}.$$

It will be useful to note that for all $a \geq 1$ we have that ϱ^a is convex, so that

$$\varrho^a(x, y) \lesssim \varrho^a(x - \bar{x}, y - \bar{y}) + \varrho^a(\bar{x}, \bar{y}) \qquad \text{for all} \quad (x, y), (\bar{x}, \bar{y}) \in \mathbb{R}^2. \quad (3.3)$$

Notice that by (3.3) and Young's inequality

$$\begin{aligned} \|\varrho^2\phi\|_\infty &\lesssim \|\varrho^2 K_\alpha * \phi^2\|_\infty + \|K_\alpha * \varrho^2\phi^2\|_\infty \\ &\lesssim \|\varrho^2 K_\alpha\|_\infty \|\phi\|^2_{L^2(\mathbb{R}^2)} + \|K_\alpha\|_{L^q(\mathbb{R}^2)} \|\varrho^2\phi^2\|_{L^{q'}(\mathbb{R}^2)} \end{aligned}$$

for some $1 \leq q, q' \leq \infty$ with $1 = \frac{1}{q} + \frac{1}{q'}$, so that the statement of Theorem 2 is proved provided that

(A) $\varrho^2 K_\alpha \in L^\infty(\mathbb{R}^2)$
(B) there exists $1 \leq q \leq \infty$ such that $K_\alpha \in L^q(\mathbb{R}^2)$ and $\varrho^2\phi^2 \in L^{q'}(\mathbb{R}^2)$, where q' is the dual conjugate of q.

Before studying the properties of the kernel function K_α, we state the following two lemmata, which yield some a priori regularity of lump solutions in the energy space.

Lemma 3 *Any solution ϕ of (3.2) in the energy space $X_{\frac{\alpha}{2}}$ satisfies $\phi \in L^r(\mathbb{R}^2)$ for all $2 \leq r < \infty$ and $\phi \in H^\infty(\mathbb{R}^2)$. In particular, ϕ is uniformly continuous and decays to zero at infinity.*

Proof Let us start by repeating the Hörmander–Mikhilin multiplier theorem [30], which states that if $f : \mathbb{R}^2 \to \mathbb{R}$ is a function, which is smooth outside the origin and

$$\xi \mapsto \xi_1^{k_1} \xi_2^{k_2} \frac{\mathrm{d}^k}{\mathrm{d}\xi_1^{k_1} \mathrm{d}\xi_2^{k_2}} f(\xi)$$

is bounded on $\mathbb{R}^2$ for all $k_1, k_2 \in \{0, 1\}$ with $k = k_1 + k_2 \in \{0, 1, 2\}$, then f is a Fourier multiplier on $L^p(\mathbb{R}^2)$ for all $1 < p < \infty$, i.e. the operator T_f defined by

$T_f g = \mathcal{F}^{-1}\left(f\hat{g}\right) = \mathcal{F}^{-1}(f) * g$ is bounded on $L^p(\mathbb{R}^2)$. By the Hörmander–Mikhilin multiplier theorem, it is easy to check that the functions

$$\xi \mapsto m_\alpha(\xi), \qquad \xi \mapsto |\xi_1|^\alpha m_\alpha(\xi), \qquad \xi \mapsto \xi_2 m_\alpha(\xi)$$

are Fourier multipliers on $L^p(\mathbb{R}^2)$ for $1 < p < \infty$. Let $\phi \in X_{\frac{\alpha}{2}}$. Due to Proposition 1 (ii), we have that $\phi \in L^3(\mathbb{R}^2)$, which implies $\phi^2 \in L^{\frac{3}{2}}(\mathbb{R}^2)$. Since

$$\phi = \frac{1}{2}\mathcal{F}^{-1}(m_\alpha) * \phi^2,\ \mathrm{D}_x^\alpha \phi = \frac{1}{2}\mathcal{F}^{-1}(|\xi_1|^\alpha m_\alpha) * \phi^2,\ \phi_y = -\frac{\mathrm{i}}{2}\mathcal{F}^{-1}(\xi_2 m_\alpha) * \phi^2,$$

we find

$$\phi, \mathrm{D}_x^\alpha \phi, \phi_y \in L^{\frac{3}{2}}(\mathbb{R}^2).$$

In particular, ϕ belongs to the anisotropic Sobolev space $W^{\vec{\alpha},\frac{3}{2}}(\mathbb{R}^2)$ for $\vec{\alpha} = (\alpha, 1)$, where

$$W^{\vec{\alpha},q}(\mathbb{R}^2) := \{\phi \in L^q(\mathbb{R}^2) \mid \mathrm{D}_x^{\alpha_1}\phi, \mathrm{D}_y^{\alpha_2}\phi \in L^q(\mathbb{R}^2)\}.$$

We use the following anisotropic Gagliardo–Nirenberg inequality for fractional derivatives [13, Theorem 1.1]: If $\phi \in W^{\vec{\alpha},q}(\mathbb{R}^2)$ with $A := \frac{1}{q}\left(\frac{1}{\alpha_1} + \frac{1}{\alpha_2}\right) - 1 > 0$ and $M := 1 + (\frac{1}{p} - \frac{1}{q})\left(\frac{1}{\alpha_1} + \frac{1}{\alpha_2}\right) > 0$, then

$$\|\phi\|_{L^r(\mathbb{R}^2)} \lesssim \|\phi\|_{L^p(\mathbb{R}^2)}^{1-\theta} \|\mathrm{D}_x^{\alpha_1}\phi\|_{L^q(\mathbb{R}^2)}^{\theta_1} \|\mathrm{D}_y^{\alpha_2}\phi\|_{L^q(\mathbb{R}^2)}^{\theta_2}, \tag{3.4}$$

for all

$$p \le r < r_* := \frac{1}{A}\left(\frac{1}{\alpha_1} + \frac{1}{\alpha_2}\right),$$

where $\theta = \theta_1 + \theta_2$ and $\theta_i = (\frac{1}{p} - \frac{1}{r})(\alpha_i M)^{-1}$. Applied to the situation at hand, we can choose $p = 2$, $q = \frac{3}{2}$ and $\vec{\alpha} = (\alpha, 1)$ for $\alpha = \frac{4}{5+4\varepsilon}$ with $\varepsilon > 0$ arbitrarily small, which yields $A = \frac{1}{2} + \frac{2}{3}\varepsilon$ and $M = \frac{5}{8} - \frac{1}{6}\varepsilon$. Due to (3.4) we find that

$$\phi \in L^r(\mathbb{R}^2) \qquad \text{for all} \quad 2 \le r < \frac{9}{2}.$$

Repeating the same argument for $\phi \in L^{\frac{9}{2}-2\varepsilon}(\mathbb{R}^2)$ with $\phi^2 \in L^{\frac{9-4\varepsilon}{4}}(\mathbb{R}^2)$, we find that $\phi \in W^{(\alpha,1),\frac{9-4\varepsilon}{4}}(\mathbb{R}^2)$ and again by the fractional Gagliardo–Nirenberg inequality for $p = 2$, $q = \frac{9-4\varepsilon}{4}$, $\vec{\alpha} = (\alpha, 1)$ for $\alpha = \frac{4}{5+4\varepsilon}$, we obtain that $A = \frac{8\varepsilon}{9-4\varepsilon}$ and $M = \frac{9}{8} + \frac{\varepsilon}{2}\frac{4\varepsilon+7}{4\varepsilon-9}$, so that

$$\phi \in L^r(\mathbb{R}^2) \qquad \text{for all} \quad 2 \le r < \infty, \tag{3.5}$$

by letting $\varepsilon \to 0$. This proves the first assertion. The relation (3.5) implies by the Fourier multiplier theorem that

$$\phi, \mathrm{D}_x^\alpha \phi, \phi_y \in L^r(\mathbb{R}^2) \qquad \text{for all} \quad 2 \le r < \infty.$$

Next, we aim to bootstrap the smoothness. By Hölder's inequality it is clear also that $(\phi^2)_y \in L^r(\mathbb{R}^2)$ for all $2 \le r < \infty$. Due to the Leibniz rule for fractional derivatives (see e.g. [17, Theorem 7.6.1]) and Hölder's inequality we can estimate

$$\begin{aligned}
\|\mathrm{D}_x^\alpha \phi^2\|_{L^r(\mathbb{R}^2)}^r &= \int_{\mathbb{R}} \|\mathrm{D}_x^\alpha \phi(\cdot, y)\|_{L^r(\mathbb{R})}^r \,\mathrm{d}y \\
&\lesssim \int_{\mathbb{R}} \|\mathrm{D}_x^\alpha \phi(\cdot, y)\|_{L^{2r}(\mathbb{R})}^r \|\phi(\cdot, y)\|_{L^{2r}(\mathbb{R})}^r \,\mathrm{d}y \\
&\le \left(\int_{\mathbb{R}} \|\mathrm{D}_x^\alpha \phi(\cdot, y)\|_{L^{2r}(\mathbb{R})} \,\mathrm{d}y \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}} \|\phi(\cdot, y)\|_{L^{2r}(\mathbb{R})}^{2r} \,\mathrm{d}y \right)^{\frac{1}{2}} \\
&= \|\mathrm{D}_x^\alpha \phi\|_{L^{2r}(\mathbb{R}^2)}^{\frac{r}{2}} \|\phi\|_{L^{2r}(\mathbb{R}^2)}^{\frac{r}{2}},
\end{aligned}$$

which yields that also $\mathrm{D}_x^\alpha \phi^2 \in L^r(\mathbb{R}^2)$ for all $2 \le r < \infty$. This can be used to bootstrap the smoothness of ϕ, by using

$$\mathrm{D}_x^\alpha \phi = \frac{1}{2} K_\alpha * \mathrm{D}_x^\alpha \phi^2 \qquad \text{and} \qquad \phi_y = \frac{1}{2} K_\alpha * (\phi^2)_y.$$

Reiterating the argument yields

$$\phi, \mathrm{D}_x^\alpha \phi, \phi_y, \mathrm{D}_x^{2\alpha} \phi, \mathrm{D}_x^\alpha \phi_y, \phi_{yy} \in L^r(\mathbb{R}^2) \qquad \text{for all} \quad 2 \le r < \infty.$$

and eventually $\mathrm{D}_x^k \phi \in L^r(\mathbb{R}^2)$ for all $2 \le r < \infty$ and $k \in \mathbb{N}$, which implies that $\phi \in H^\infty(\mathbb{R}^2)$. Eventually, since $H^\infty(\mathbb{R}^2)$ is embedded into the space of uniformly continuous functions on $\mathbb{R}^2$ and ϕ is $L^2(\mathbb{R}^2)$-integrable, we deduce that ϕ decays to zero at infinity. □

Lemma 4 *Any solution ϕ of (3.1) in the energy space $X_{\frac{\alpha}{2}}$ satisfies*

$$\int_{\mathbb{R}^2} (x^2 + y^2)(|\phi_x|^2 + |\phi_y|^2 + |\mathrm{D}_x^{\frac{\alpha}{2}} \phi_x|^2) \,\mathrm{d}(x, y) < \infty.$$

Proof The proof follows essentially the lines in [10, Lemma 3.1]. Here, we proceed formally by omitting the truncation function at infinity. First, let us multiply (3.1) by $x^2\phi$ and integrate over $\mathbb{R}^2$. Then

$$0 = \int_{\mathbb{R}^2} x^2 \phi \left(-\phi_{xx} - \phi_{yy} - \mathrm{D}_x^\alpha \phi_{xx} + \frac{1}{2} \left(\phi^2\right)_{xx} \right) \mathrm{d}(x, y).$$

Using integration by parts we find that

$$\int_{\mathbb{R}^2} x^2\phi\left(-\phi_{xx}-\phi_{yy}+\frac{1}{2}\left(\phi^2\right)_{xx}\right)\mathrm{d}(x,y)=\int_{\mathbb{R}^2} x^2\left(\phi_x^2+\phi_y^2-\phi\phi_x^2\right) - \phi^2+\frac{2}{3}\phi^3\,\mathrm{d}(x,y).$$

In view of Lemma 8, the nonlocal part can be written as

$$\begin{aligned}-\int_{\mathbb{R}^2} x^2\phi \mathrm{D}_x^\alpha\phi_{xx}\,d(x,y) &= -\int_{\mathbb{R}^2} x^2\phi_{xx}\mathrm{D}_x^\alpha\phi\,\mathrm{d}(x,y)-\int_{\mathbb{R}^2}4x\phi_x\mathrm{D}_x^\alpha\phi\,d(x,y)\\ &\quad -\int_{\mathbb{R}^2}2\phi\mathrm{D}_x^\alpha\phi\,\mathrm{d}(x,y)\\ &=\int_{\mathbb{R}^2} x^2\left(\mathrm{D}^{\frac{\alpha}{2}}\phi_x\right)^2\mathrm{d}(x,y)-\frac{1}{4}(\alpha+2)^2\\ &\quad\times\int_{\mathbb{R}^2}\left(\mathrm{D}^{\frac{\alpha}{2}}\phi\right)^2\mathrm{d}(x,y).\end{aligned}$$

Adding the above equalities we obtain that

$$\int_{\mathbb{R}^2} x^2\left(\phi_x^2+\phi_y^2+\left(\mathrm{D}^{\frac{\alpha}{2}}\phi_x\right)^2\right)\mathrm{d}(x,y)=\int_{\mathbb{R}^2} x^2\phi\phi_x^2+\phi^2-\frac{2}{3}\phi^3 +\frac{1}{4}(\alpha+2)^2\left(\mathrm{D}^{\frac{\alpha}{2}}\phi\right)^2\mathrm{d}(x,y) \tag{3.6}$$

Multiplying (3.1) by $y^2\phi$ instead yields

$$\int_{\mathbb{R}^2} y^2\phi\left(-\phi_{xx}-\phi_{yy}-\mathrm{D}_x^\alpha\phi_{xx}+\frac{1}{2}\left(\phi^2\right)_{xx}\right)\mathrm{d}(x,y)=0.$$

Again, using integration by parts, we find that

$$\int_{\mathbb{R}^2} y^2\left(\phi_x^2+\phi_y^2+\left(\mathrm{D}_x^{\frac{\alpha}{2}}\phi_x\right)^2\right)\mathrm{d}(x,y)=\int_{\mathbb{R}^2}\phi^2+y^2\phi\phi_x^2\,\mathrm{d}(x,y). \tag{3.7}$$

Adding (3.6) and (3.7), while keeping in mind that $\phi\in X_{\frac{\alpha}{2}}$, we can estimate

$$\int_{\mathbb{R}^2}(x^2+y^2)(|\phi_x|^2+|\phi_y|^2+|\mathrm{D}_x^{\frac{\alpha}{2}}\phi_x|^2)\,\mathrm{d}(x,y)\lesssim 1+\int_{\mathbb{R}^2}(x^2+y^2)\phi\phi_x^2\,\mathrm{d}(x,y).$$

Using that ϕ is continuous and tends to zero at infinity (see Lemma 3), there exists $R>0$ such that $\phi(x,y)\le\frac{1}{2}$ for $|(x,y)|\ge R$ and we conclude

$$\int_{\mathbb{R}^2}(x^2+y^2)(|\phi_x|^2+|\phi_y|^2+|\mathrm{D}_x^{\frac{\alpha}{2}}\phi_x|^2)\lesssim_R 1.$$

□

Properties of the kernel function K_α. We will first concentrate on the regularity properties of K_α.

Lemma 5 $m_\alpha \in L^p(\mathbb{R}^2)$ *if and only if* $p > \frac{2}{\alpha} + \frac{1}{2}$.

Proof It is clear that $m_\alpha \in L^\infty(\mathbb{R}^2)$. Let us compute

$$\begin{aligned}\|m_\alpha\|^p_{L^p(\mathbb{R}^2)} &= \int_{\mathbb{R}^2} \frac{1}{\left(1+|\xi_1|^\alpha + \frac{\xi_2^2}{\xi_1^2}\right)^p} \, \mathrm{d}(x,y) \\ &= \int_{\mathbb{R}} \frac{1}{(1+|\xi_1|^\alpha)^p} \int_{\mathbb{R}} \frac{1}{\left(1+\frac{\xi_2^2}{(1+|\xi_1|^\alpha)\xi_1^2}\right)^p} \, \mathrm{d}\xi_2 \, \mathrm{d}\xi_1 \\ &= \int_{\mathbb{R}} \frac{|\xi_1|}{(1+|\xi_1|^\alpha)^{p-\frac{1}{2}}} \, \mathrm{d}\xi_1 \int_{\mathbb{R}} \frac{1}{\left(1+z^2\right)^p} \, \mathrm{d}z,\end{aligned}$$

where we used the change of variables $z = \frac{\xi_2}{|\xi_1|(1+|\xi_1|^\alpha)^{\frac{1}{2}}}$. Since the second integral above is clearly convergent for any $p > \frac{1}{2}$, we find that $m_\alpha \in L^p(\mathbb{R}^2)$ if and only if

$$p > \frac{2}{\alpha} + \frac{1}{2}.$$

□

Remark 3 The above lemma implies that $m_\alpha \in L^2(\mathbb{R}^2)$ if and only if $\alpha > \frac{4}{3}$, which is the L^2-critical exponent. In this case it follows immediately, that also $K_\alpha \in L^2(\mathbb{R}^2)$ and the proof of Theorem 2 can be done essentially by following the lines in [10]. In the supercritical case $\frac{4}{5} < \alpha \leq \frac{4}{3}$, which in particular includes the Benjamin–Ono KP equation for $\alpha = 1$, the symbol m_α belongs to an L^p-space with $p > 2$ so that the integrability properties of the kernel K_α are a priori not clear.

Lemma 6 *The kernel function* K_α *is smooth outside the origin.*

Proof Let $\chi : \mathbb{R}^2 \to \mathbb{R}$ be a compactly supported, radial, smooth function with $\chi(0,0) = 1$. Set $\bar{m}_\alpha := \chi m_\alpha$. Then $\bar{m}_\alpha$ has compact support and $\mathcal{F}^{-1}(\bar{m}_\alpha)$ is real analytic. Now, set $\tilde{m}_\alpha := (1-\chi)m_\alpha$. Then $\tilde{m}_\alpha$ is smooth. Let us fix $(x_0, y_0) \neq (0,0)$ and let $\psi : \mathbb{R}^2 \to \mathbb{R}$ be a compactly supported, smooth function with $\psi(x,y) = 1$ in an arbitrarily small neighborhood of (x_0, y_0) and $\psi(0,0) = 0$. Then also

$$\Psi_k(x,y) = |(x,y)|^{-2k}\psi$$

is smooth and compactly supported. Notice that $\hat{\Psi}_k = -\Delta^{-k}\hat{\psi}$ and

$$\tilde{m}_\alpha * \hat{\psi} = -\tilde{m}_\alpha * \Delta^k \hat{\Psi}_k = -(\Delta^k \tilde{m}_\alpha * \hat{\Psi}_k).$$

Since Ψ_k is smooth with compact support, we know that $\hat{\Psi}_k \in \mathcal{S}(\mathbb{R}^2)$. Furthermore $\Delta^k \tilde{m}_\alpha$ is smooth with $\Delta^k \tilde{m}_\alpha(\xi) \lesssim \frac{1}{|\xi|^{\alpha+2k}}$ for $|\xi| \to \infty$. Since the convolution of two integrable, smooth functions is smooth and decays at least as fast as the function with the lower decay, we deduce that $\tilde{m}_\alpha * \hat{\psi}$ is smooth and decays at least as $\frac{1}{|\cdot|^{\alpha+2k}}$ at infinity for an arbitrary choice of $k \in \mathbb{N}$. In particular, $\mathcal{F}^{-1}(\tilde{m}_\alpha * \hat{\psi}) = \mathcal{F}^{-1}(\tilde{m}_\alpha)\psi$ is smooth, which yields that $\mathcal{F}^{-1}(\tilde{m}_\alpha)$ is smooth outside the origin. We conclude that

$$K_\alpha = \mathcal{F}^{-1}(\bar{m}_\alpha) + \mathcal{F}^{-1}(\tilde{m}_\alpha)$$

is smooth outside the origin. □

Let us now investigate the behavior of K_α at infinity. We show that the decay is quadratic, independently of the value of $\alpha > 0$.

Proposition 6 *For any $\alpha > 0$, we have that $\varrho^2 K_\alpha$ belongs to $L^\infty(\mathbb{R}^2)$.*

Proof We have $K_\alpha = \mathcal{F}^{-1}(m_\alpha)$, so that

$$\begin{aligned} K_\alpha(x, y) &= \int_{\mathbb{R}^2} \frac{\xi_1^2}{\xi_1^2 + \xi_2^2 + |\xi_1|^{\alpha+2}} e^{\mathrm{i}x\xi_1 + \mathrm{i}y\xi_2} \,\mathrm{d}\xi_1 \,\mathrm{d}\xi_2 \\ &= \int_{\mathbb{R}} \frac{|\xi|}{(1 + |\xi|^\alpha)^{\frac{1}{2}}} e^{-|y||\xi|(1+|\xi|^\alpha)^{\frac{1}{2}}} e^{\mathrm{i}x\xi} \,\mathrm{d}\xi, \end{aligned}$$

where we used that

$$\mathcal{F}\left(\frac{1}{a^2 + (\cdot)^2}\right)(y) = \int_{\mathbb{R}} \frac{1}{a^2 + \xi_2^2} e^{-\mathrm{i}\xi_2 y} \,\mathrm{d}\xi_2 = \frac{1}{a} e^{-a|y|}$$

and $a^2 = \xi_1^2 + |\xi_1|^{\alpha+2}$. Let us consider the case where $\xi \geq 0$ (the proof works similarly for $\xi < 0$). Assume for the moment that $y \neq 0$. Setting

$$K_\alpha^+(x, y) := \int_0^\infty \frac{\xi}{(1 + \xi^\alpha)^{\frac{1}{2}}} e^{-|y|\xi(1+\xi^\alpha)^{\frac{1}{2}}} e^{\mathrm{i}x\xi} \,\mathrm{d}\xi,$$

we can write

$$K_\alpha^+(x, y) = \int_0^\infty \frac{\xi}{(1 + \xi^\alpha)^{\frac{1}{2}}} \frac{1}{G'(\xi)} \frac{d}{d\xi} \left(e^{G(\xi)}\right) \mathrm{d}\xi,$$

where $G(\xi) := \mathrm{i}x\xi - |y|\xi(1 + \xi^\alpha)^{\frac{1}{2}}$. Using integration by parts, we obtain

$$K_\alpha^+(x, y) = -\int_0^\infty \frac{\mathrm{d}}{\mathrm{d}\xi} \left(\frac{\xi}{(1 + \xi^\alpha)^{\frac{1}{2}}} \frac{1}{G'(\xi)}\right) e^{G(\xi)} \,\mathrm{d}\xi.$$

Applying again integration by parts, we find

$$
\begin{aligned}
K_\alpha^+(x,y) = &-\left[\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\frac{\xi}{(1+\xi^\alpha)^{\frac{1}{2}}}\frac{1}{G'(\xi)}\right)\frac{1}{G'(\xi)}e^{G(\xi)}\right]_0^\infty \\
&+\int_0^\infty \frac{\mathrm{d}}{\mathrm{d}\xi}\left(\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\frac{\xi}{(1+\xi^\alpha)^{\frac{1}{2}}}\frac{1}{G'(\xi)}\right)\frac{1}{G'(\xi)}\right)e^{G(\xi)}\,\mathrm{d}\xi \\
= &\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\frac{\xi}{(1+\xi^\alpha)^{\frac{1}{2}}}\frac{1}{G'(\xi)}\right)\frac{1}{G'(\xi)}\Big|_{\xi=0} \\
&+\int_0^\infty \frac{\mathrm{d}}{\mathrm{d}\xi}\left(\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\frac{\xi}{(1+\xi^\alpha)^{\frac{1}{2}}}\frac{1}{G'(\xi)}\right)\frac{1}{G'(\xi)}\right)e^{G(\xi)}\,\mathrm{d}\xi.
\end{aligned}
$$

In order to lighten the notation, we set

$$F(\xi) := \frac{\mathrm{d}}{\mathrm{d}\xi}\left(\frac{\xi}{(1+\xi^\alpha)^{\frac{1}{2}}}\frac{1}{G'(\xi)}\right)\frac{1}{G'(\xi)},$$

so that

$$K_\alpha^+(x,y) = F(0) + \int_0^\infty F'(\xi)e^{G(\xi)}\,\mathrm{d}\xi.$$

Using Lemma 9, we find

$$|K_\alpha^+(x,y)| \le \frac{1}{x^2+y^2}.$$

We are left to consider the case when $y = 0$, that is

$$K_\alpha(x,0) = \int_{\mathbb{R}} \frac{|\xi|}{(1+|\xi|^\alpha)^{\frac{1}{2}}}e^{\mathrm{i}x\xi}\,\mathrm{d}\xi.$$

Notice that $x^2K_\alpha(x,0) = -\mathcal{F}^{-1}\left(\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}\frac{|\xi|}{(1+|\xi|^\alpha)^{\frac{1}{2}}}\right)$ and

$$\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}\frac{|\xi|}{(1+|\xi|^\alpha)^{\frac{1}{2}}} = 2\delta_0(\xi) + g(\xi),$$

where δ_0 denotes the delta distribution centered at zero and $g \in L^1(\mathbb{R})$. Thus $x \mapsto x^2K_\alpha(x,0)$ belongs to $L^\infty(\mathbb{R})$. □

In order to determine the L^p-regularity of K_α, it is left to investigate the behaviour of the kernel function close to the origin. To do so, we will use that $|\nabla m_\alpha| \lesssim h_\alpha$, where

$$h_\alpha(\xi_1,\xi_2) := \frac{\xi_1}{|\xi|^2 + \xi_1^{\alpha+2}}. \tag{3.8}$$

Notice also that $\widehat{\partial_x^{-1} K_\alpha}(\xi_1,\xi_2) = -\mathrm{i}\, h(\xi_1,\xi_2)$ and (3.2) can be written as

$$\phi = -\frac{\mathrm{i}}{2} H_\alpha * (\phi^2)_x, \qquad \hat{H}_\alpha(\xi_1,\xi_2) = h_\alpha(\xi_1,\xi_2). \tag{3.9}$$

Lemma 7 *(The symbol h_α) We have that*
a) $h_\alpha \in L^p(\mathbb{R}^2)$ *if and only if* $\frac{1}{2} + \frac{3}{2(1+\alpha)} < p < 2$ *and*

$$H_\alpha \in L^{p'}(\mathbb{R}^2) \quad \textit{for} \;\; 2 < p' < \frac{4+\alpha}{2-\alpha}.$$

b) $\varrho H_\alpha \in L^\infty(\mathbb{R}^2)$.

Proof Similar as in the proof of Lemma 5 we compute

$$\begin{aligned}
\|h_\alpha\|^p_{L^p(\mathbb{R}^2)} &= \int_{\mathbb{R}^2} \frac{1}{\left(|\xi_1| + |\xi_1|^{\alpha+1} + \frac{\xi_2^2}{|\xi_1|}\right)^p} \,\mathrm{d}(\xi_1,\xi_2)\\
&= \int_{\mathbb{R}} \frac{1}{|\xi_1|^p\,(1+|\xi_1|^\alpha)^p} \int_{\mathbb{R}} \frac{1}{\left(1 + \frac{\xi_2^2}{\xi_1^2(1+|\xi_1|^\alpha)}\right)^p} \,\mathrm{d}\xi_2\,\mathrm{d}\xi_1\\
&= \int_{\mathbb{R}} \frac{1}{|\xi_1|^{p-1}\,(1+|\xi_1|^\alpha)^{p-\frac{1}{2}}} \,\mathrm{d}\xi_1 \int_{\mathbb{R}} \frac{1}{(1+z^2)^p}\,\mathrm{d}z,
\end{aligned}$$

where we used the change of variables $z = \frac{\xi_2}{|\xi_1|(1+|\xi_1|^\alpha)^{\frac{1}{2}}}$. Since the last integral above is bounded for all $p > 1$, we find that $h_\alpha \in L^p(\mathbb{R}^2)$ if and only if

$$\frac{1}{2} + \frac{3}{2(1+\alpha)} < p < 2.$$

Since the Fourier transform is a bounded function from $L^p(\mathbb{R}^2)$ to $L^{p'}(\mathbb{R}^2)$ for $p \in [1,2]$ and p' being the dual conjugate to p, we obtain immediately that

$$H_\alpha \in L^{p'}(\mathbb{R}^2) \quad \text{for} \;\; 2 < p' < \frac{4+\alpha}{2-\alpha}.$$

Thereby, part (a) is proved. In order to prove part (b) we proceed as in the proof of Proposition 6. We have that

$$H_\alpha(x,y) = \int_{\mathbb{R}^2} \frac{\xi_1}{\xi_1^2 + \xi_2^2 + |\xi|^{\alpha+2}} e^{\mathrm{i}x\xi_1 + \mathrm{i}y\xi_2}\,\mathrm{d}(\xi_1,\xi_2)$$

$$
\begin{aligned}
&= \int_{\mathbb{R}} \xi_1 e^{\mathrm{i}x\xi_1} \int_{\mathbb{R}} \frac{1}{\xi_1^2 + \xi_2^2 + |\xi_1|^{\alpha+2}} e^{\mathrm{i}\xi_2 y} \, \mathrm{d}\xi_2 \, \mathrm{d}\xi_1 \\
&= \int_{\mathbb{R}} \frac{\xi}{|\xi|(1+|\xi|^\alpha)^{\frac{1}{2}}} e^{\mathrm{i}x\xi - |\xi|(1+|\xi|^\alpha)^{\frac{1}{2}}|y|} \, \mathrm{d}\xi \\
&= \int_{\mathbb{R}} \operatorname{sgn}(\xi) \frac{1}{(1+|\xi|^\alpha)^{\frac{1}{2}}} e^{\mathrm{i}x\xi - |\xi|(1+|\xi|^\alpha)^{\frac{1}{2}}|y|} \, \mathrm{d}\xi.
\end{aligned}
$$

Let us consider the positive part of the integral, the negative part can be estimated analogously. Assume for the moment that $y \neq 0$ and set

$$
H_\alpha^+(x, y) := \int_0^\infty \frac{1}{(1+|\xi|^\alpha)^{\frac{1}{2}}} e^{\mathrm{i}x\xi - |\xi|(1+|\xi|^\alpha)^{\frac{1}{2}}|y|} \, \mathrm{d}\xi.
$$

With $E(\xi) := \frac{1}{(1+\xi^\alpha)^{\frac{1}{2}}} \frac{1}{G'(\xi)}$, we obtain after integration by parts

$$
H_\alpha^+(x, y) = -E(0) - \int_0^\infty E'(\xi) e^{G(\xi)} \, \mathrm{d}\xi,
$$

where $G(\xi) = \mathrm{i}x\xi - |\xi|(1+\xi^\alpha)^{\frac{1}{2}}|y|$. In view of Lemma 10 we find that

$$
|H_\alpha^+(x, y)| \lesssim \frac{1}{\sqrt{x^2 + y^2}}. \tag{3.10}
$$

If $y = 0$, we have

$$
H_\alpha(x, 0) = \int_{\mathbb{R}} \frac{\xi}{|\xi|(1+|\xi|^\alpha)^{\frac{1}{2}}} e^{\mathrm{i}x\xi} \, \mathrm{d}\xi.
$$

Notice that $\mathrm{i}x H_\alpha(x, y) = -\mathcal{F}^{-1}\left(\frac{\mathrm{d}}{\mathrm{d}\xi} \frac{\xi}{|\xi|(1+|\xi|^\alpha)^{\frac{1}{2}}}\right)$ and

$$
\frac{\mathrm{d}}{\mathrm{d}\xi} \frac{\xi}{|\xi|(1+|\xi|^\alpha)^{\frac{1}{2}}} = \frac{1}{(1+|\xi|^\alpha)^{\frac{1}{2}}} \delta_0(\xi) + g(\xi),
$$

where δ_0 denotes the delta distribution centered at zero and $g \in L^1(\mathbb{R})$. We deduce that $x \mapsto |x| H_\alpha(x, 0)$ is a bounded function. Together with (3.10) this proves the claim that $\varrho H_\alpha \in L^\infty(\mathbb{R}^2)$. □

Proposition 7 *The kernel function K_α satisfies the regularity*

$$
K_\alpha \in L^r(\mathbb{R}^2) \quad \textit{for} \quad 1 < r < \frac{8 + 2\alpha}{8 - \alpha}.
$$

Proof We know already from Lemma 6 and Proposition 6 that K_α is smooth outside the origin and $\varrho^2 K_\alpha \in L^\infty(\mathbb{R}^2)$. Introducing a smooth truncation function $\vartheta : \mathbb{R}_+ \to \mathbb{R}_+$, which is compactly supported in a neighborhood of zero, denoted by $B \subset \mathbb{R}^2$, with $\vartheta(0) = 1$, we find that

$$(1 - \vartheta(\varrho))K_\alpha \in L^s(\mathbb{R}^2) \qquad \text{for all} \quad s > 1. \tag{3.11}$$

In order to determine the regularity of K_α close to zero, recall that $|\nabla m_\alpha| \lesssim |h_\alpha|$, where h_α defined in (3.8) so that

$$|\nabla m_\alpha| \in L^q(\mathbb{R}^2) \qquad \text{for} \qquad \frac{1}{2} + \frac{3}{2(1+\alpha)} < q < 2,$$

due to Lemma 7 (a). Now, we use that the Fourier transformation is a bounded operator from $L^p(\mathbb{R}^2)$ to $L^{p'}(\mathbb{R}^2)$ when $p \in [1, 2]$ and p' is the dual conjugate of p and obtain that

$$\begin{aligned} \|\varrho K_\alpha\|_{L^{q'}(\mathbb{R}^2)} &\lesssim \|xK\|_{L^{q'}(\mathbb{R}^2)} + \|yK\|_{L^{q'}(\mathbb{R}^2)} \\ &\le \|\partial_{\xi_1} m_\alpha\|_{L^q(\mathbb{R}^2)} + \|\partial_{\xi_2} m_\alpha\|_{L^q(\mathbb{R}^2)} \\ &\lesssim 2\||\nabla m_\alpha|\|_{L^q(\mathbb{R}^2)}, \end{aligned}$$

so that

$$\varrho K_\alpha \in L^{q'}(\mathbb{R}^2) \qquad \text{for} \qquad 2 < q' < \frac{4+\alpha}{2-\alpha},$$

and in fact $\varrho K_\alpha \in L^s(B)$ for $1 \le s < \frac{4+\alpha}{2-\alpha}$, by Hölder's inequality and the boundedness of B. Then, we estimate

$$\|\vartheta(\varrho)K_\alpha\|_{L^r(\mathbb{R}^2)} \lesssim \|\varrho^{-1}\|_{L^t(B)}\|\varrho K_\alpha\|_{L^s(B)},$$

where $\frac{1}{r} = \frac{1}{t} + \frac{1}{s}$. Since $\varrho^{-1} \in L^t(B)$ if and only if $1 \le t < 2$, we conclude that

$$\vartheta(\varrho)K_\alpha \in L^r(\mathbb{R}^2) \qquad \text{for} \quad 1 \le r < \frac{8+2\alpha}{8-\alpha}.$$

In view of (3.11) we deduce that

$$K_\alpha \in L^r(\mathbb{R}^2) \qquad \text{for} \quad 1 < r < \frac{8+2\alpha}{8-\alpha}.$$

□

A non-optimal decay rate. First notice that ϕ inherits the integrability properties of K_α, since

$$\|\phi\|_{L^r(\mathbb{R}^2)} = \frac{1}{2}\|K_\alpha * \phi^2\|_{L^r(\mathbb{R}^2)} \lesssim \|K_\alpha\|_{L^r(\mathbb{R}^2)}\|\phi^2\|_{L^1(\mathbb{R}^2)} \lesssim \|K_\alpha\|_{L^r(\mathbb{R}^2)}\|\phi\|^2_{L^2(\mathbb{R}^2)},$$

by the L^2-integrability of ϕ. Interpolating between the boundedness of ϕ, which is due to Lemma 3, and the L^r-integrability of ϕ for $1 < r < \frac{8+2\alpha}{8-\alpha}$, we actually find that

$$\phi \in L^p(\mathbb{R}^2) \qquad \text{for all} \quad p > 1. \tag{3.12}$$

Proposition 8 *(A priori decay estimate) If ϕ is a solution of (3.1) in the energy space $X_{\frac{\alpha}{2}}$, then*

$$\varrho\phi \in L^\infty(\mathbb{R}^2).$$

Proof Recall from (3.9) that

$$\phi = -\frac{\mathrm{i}}{2} H_\alpha * (\phi^2)_x$$

so that by Young's inequality

$$\begin{aligned}\|\varrho\phi\|_\infty &\le \|\varrho H * \phi\phi_x\|_\infty + \|H * \varrho\phi\phi_x\|_\infty \\ &\lesssim \|\varrho H\|_\infty \|\phi\|_{L^2(\mathbb{R})} \|\phi_x\|_{L^2(\mathbb{R})} + \|H\|_{L^{q'}(\mathbb{R}^2)} \|\varrho\phi_x\|_{L^2(\mathbb{R}^2)} \|\phi\|_{L^s(\mathbb{R}^2)},\end{aligned}$$

where $\frac{1}{s} + \frac{1}{2} = \frac{1}{q}$ for $\frac{1}{2} + \frac{3}{2(1+\alpha)} < q < 2$ and q' being the dual of q. Now, the statement follows from Lemma 7 and Lemma 4. □

Proposition 9 *(A non-optimal decay rate) If ϕ is a solitary solution of (3.1), then*

$$\varrho^{1+\delta}\phi \in L^\infty(\mathbb{R}^2)$$

for any $0 \le \delta < 1$.

Proof We use the regularity in Proposition 8 to improve the decay rate by estimating

$$\|\varrho^{1+\delta}\phi\|_\infty \lesssim \|\varrho^{1+\delta} K_\alpha * \phi^2\|_\infty + \|K_\alpha * \varrho^{1+\delta}\phi^2\|_\infty,$$

where we also used the convexity of $\varrho^{1+\delta}$. The first norm on the right-hand side above is clearly bounded by Young's inequality, Proposition 6 and the L^2-integrability of ϕ. For the second norm, let $\varepsilon > 0$ be a small constant so that $0 < \delta < \frac{1}{1+\varepsilon} < 1$. Using that $K_\alpha \in L^{1+\varepsilon}(\mathbb{R}^2)$ for $\varepsilon > 0$ small enough, we estimate

$$\|K_\alpha * \varrho^{1+\delta}\phi^2\|_\infty \lesssim \|K_\alpha\|_{L^{1+\varepsilon}(\mathbb{R}^2)} \|\varrho^{1+\delta}\phi^2\|_{L^{\frac{1+\varepsilon}{\varepsilon}}(\mathbb{R}^2)}.$$

Notice that

$$\|\varrho^{1+\delta}\phi^2\|^{\frac{\varepsilon}{1+\varepsilon}}_{L^{\frac{1+\varepsilon}{\varepsilon}}(\mathbb{R}^2)} = \int_{\mathbb{R}^2} |\varrho\phi|^{\frac{(1+\delta)(1+\varepsilon)}{\varepsilon}} |\phi|^{(1-\delta)\frac{1+\varepsilon}{\varepsilon}} \,\mathrm{d}(x,y)$$

$$\leq \|\varrho\phi\|_{\infty}^{\frac{(1+\delta)(1+\varepsilon)}{\varepsilon}} \int_{\mathbb{R}^2} |\phi|^{(1-\delta)\frac{1+\varepsilon}{\varepsilon}} \,\mathrm{d}(x,y).$$

By our choice of $\varepsilon > 0$, we have $(1-\delta)\frac{1+\varepsilon}{\varepsilon} > 1$, so the above norm is bounded by (3.12), which concludes the proof of the statement. □

Proof of Theorem 2

In view of the discussion at the beginning of this section and Lemma 3, we obtain our main result

$$\varrho^2\phi \in L^{\infty}(\mathbb{R}^2)$$

provided that (A) and (B) at the beginning of the section are satisfied. The statement in (A) is proved in Proposition 6, while the first part of statement (B) follows from Proposition 7, where it is shown that

$$K_\alpha \in L^r(\mathbb{R}^2) \qquad \text{for} \quad 1 < r < \frac{8+2\alpha}{8-\alpha}.$$

Now, we make use of the non-optimal decay estimate in Proposition 9 to show that indeed $\varrho^2\phi^2 \in L^{r'}(\mathbb{R}^2)$, where r' is the dual conjugate to r. For any $0 \leq \delta < 1$, we have that

$$\int_{\mathbb{R}^2} |\varrho^2\phi^2|^{r'} \,\mathrm{d}(x,y) \leq \|\varrho^{1+\delta}\phi\|_{\infty}^{\frac{2r'}{1+\delta}} \int_{\mathbb{R}^2} \phi^{2r'\frac{\delta}{1+\delta}} \,\mathrm{d}(x,y).$$

Choosing $\delta = r-1 \in (0,1)$ we find that $\frac{\delta}{1+\delta}r' = \frac{r-1}{r}r' = 1$ and the boundedness of $\varrho^2\phi^2$ in $L^{r'}(\mathbb{R}^2)$ follows from the L^2-integrability of ϕ. Hence, statement (B) is shown, which concludes the proof of Theorem 2.

Remark 4 (Benjamin–Bona–Mahony KP equation) The decay result in Theorem 2 is equally valid for lump solutions of the fractional BBM-KP equation, which is when the term $\mathrm{D}_x^\alpha u_x$ in (1.1) is replaced by $\mathrm{D}_x^\alpha u_t$.

Remark 5 (Rotation modified KP equation) Lump solutions $u(t,x,y) = \phi(x-ct,y)$ of the *rotation modified KP equation*

$$(u_t + uu_x - \beta u_{xxx})_x + u_{yy} = \gamma u,$$

where $\beta \in \mathbb{R}$ determines the type of dispersion and $\gamma > 0$ is the Coriolis parameter due to the Earth's rotation, exist for $\beta > 0$ and $c < 2\sqrt{\gamma\beta}$, cf. [7, Theorem 2.2, Remark 2.4]. They satisfy the convolution equation

$$\phi = -\frac{1}{2}K * \phi^2, \qquad \hat{K}(\xi_1,\xi_2) = m(\xi_1,\xi_2),$$

where

$$m(\xi_1, \xi_2) = \frac{\xi_1^2}{-c\xi_1^2 + \beta\xi_1^4 + \xi_2^2 + \gamma}.$$

Due to the Coriolis parameter $\gamma > 0$, the symbol m is *smooth* at the origin, which allows lump solutions to decay exponentially at infinity (cf. [8, Theorem 1.6]). If the dispersive term βu_{xxx} were replaced by the fractional term $-\beta|\mathrm{D}_x|^\alpha u_x$, which would lead to an *fractional rotation modified KP equation*, we'd expect a decay of lump solutions, which depends on α (in a similar way as we see it for the fractional KdV equation [14, 20]).

Remark 6 (Full dispersion KP equation) The *full dispersion KP equation* is given by

$$u_t - l(\mathrm{D})u_x + uu_x = 0, \qquad l(\mathrm{D}) = (1 + \beta|\mathrm{D}|^2)^{\frac{1}{2}} \left(\frac{\tanh(|\mathrm{D}|)}{|\mathrm{D}|}\right)^{\frac{1}{2}} \left(1 + \frac{\mathrm{D}_y^2}{\mathrm{D}_x^2}\right)^{\frac{1}{2}}.$$

Here $\beta > 0$ is the surface tension coefficient. Existence of lump solutions is shown in [11, 12]. In the same way as for the fKP-I equation, the transverse direction induces a discontinuity at the origin of the symbol $m(\xi_1, \xi_2) = \frac{1}{c + l(\xi_1, \xi_2)}$. Therefore, the decay of lump solutions is also at most quadratic.

To visualize the decay rate of lumps for the fKP-I equation we consider the product of numerically generated lumps with $\varrho^2(x, y) = x^2 + y^2$. Figure 7 shows x and $y-$cross sections of this product for $\alpha = 2$, $\alpha = 1.7$, and $\alpha = 1.35$. As the decay rate is quadratic, the result approaches a constant value for increasing $|x|$ and $|y|$. We observe that the behavior is similar for all α values but the aforementioned constant becomes smaller for smaller values of α.

4 Auxiliary results

Lemma 8 *(Fractional integration by parts) Let $\alpha \geq 0$. Then,*

$$\int_{\mathbb{R}} \phi \mathrm{D}_x^\alpha \phi \,\mathrm{d}x = \int_{\mathbb{R}^2} \left(\mathrm{D}^{\frac{\alpha}{2}} \phi\right)^2 \mathrm{d}x, \qquad \int_{\mathbb{R}} x\phi_x \mathrm{D}_x^\alpha \phi \, dx = \frac{\alpha - 1}{2} \int_{\mathbb{R}} \left(\mathrm{D}_x^{\frac{\alpha}{2}} \phi\right)^2 \mathrm{d}x$$

and

$$\int_{\mathbb{R}} x^2 \phi_{xx} \mathrm{D}_x^\alpha \phi \,\mathrm{d}x = -\int_{\mathbb{R}} x^2 \left(\mathrm{D}_x^{\frac{\alpha}{2}} \phi_x\right)^2 \mathrm{d}x + \tfrac{1}{4}(\alpha - 2)^2 \int_{\mathbb{R}} \left(\mathrm{D}_x^{\frac{\alpha}{2}} \phi\right)^2 \mathrm{d}x.$$

Proof The first assertion follows immediately by

$$\int_{\mathbb{R}} \phi \mathrm{D}_x^\alpha \phi \,\mathrm{d}x = \langle \phi, \overline{\mathrm{D}_x^\alpha \phi} \rangle = \langle \hat{\phi}, |\xi|^\alpha \overline{\hat{\phi}} \rangle = \langle |\xi|^{\frac{\alpha}{2}} \hat{\phi}, \overline{|\xi|^{\frac{\alpha}{2}} \hat{\phi}} \rangle = \int_{\mathbb{R}} \left(\mathrm{D}_x^{\frac{\alpha}{2}} \phi\right)^2 \mathrm{d}x.$$

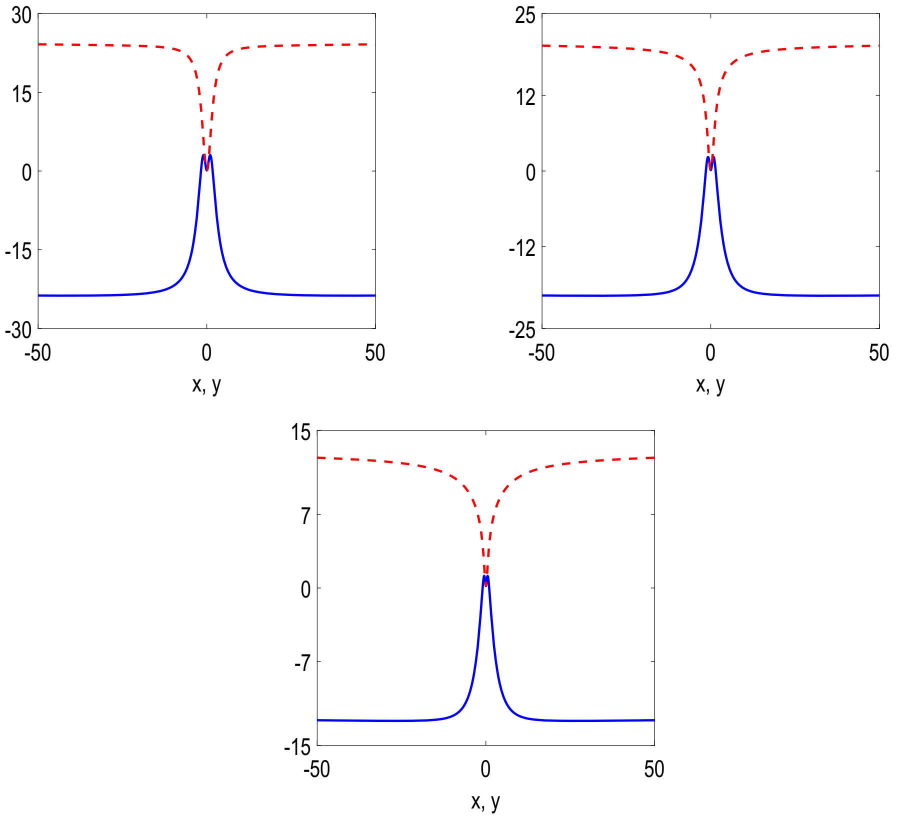

Fig. 7 The $x-$cross section (solid line) and the $y-$cross section (dashed line) of the product of numerically generated lump solutions with $\varrho^2(x, y) = (x^2 + y^2)$ for $\alpha = 2$ (top left panel), $\alpha = 1.7$ (top right panel) and $\alpha = 1.35$ (bottom panel)

For the second statement, notice first that

$$\begin{aligned}\langle \hat{\phi}_\xi, \overline{|\xi|^\alpha \xi \hat{\phi}} \rangle &= -\langle \hat{\phi}, (\alpha + 1)|\xi|^\alpha \overline{\hat{\phi}} + |\xi|^\alpha \xi \overline{\hat{\phi}_\xi} \rangle \\ &= -(\alpha + 1) \int_{\mathbb{R}^2} \left(\mathrm{D}_x^{\frac{\alpha}{2}} \phi\right)^2 \mathrm{d}x - \langle \hat{\phi}_\xi, \overline{|\xi|^\alpha \xi \hat{\phi}} \rangle,\end{aligned}$$

therefore

$$\langle \hat{\phi}_\xi, \overline{|\xi|^\alpha \xi \hat{\phi}} \rangle = -\tfrac{\alpha+1}{2} \int_{\mathbb{R}} \left(\mathrm{D}_x^{\frac{\alpha}{2}} \phi\right)^2 \mathrm{d}x, \tag{4.1}$$

which implies

$$\int_{\mathbb{R}} x \phi_x \mathrm{D}_x^\alpha \phi \,\mathrm{d}x = -\langle (\xi \hat{\phi})_\xi, |\xi|^\alpha \overline{\hat{\phi}} \rangle$$

$$= -\langle \hat{\phi}, |\xi|^{\alpha} \overline{\hat{\phi}} \rangle - \langle \hat{\phi}_{\xi}, |\xi|^{\alpha} \xi \overline{\hat{\phi}} \rangle$$
$$= -\int_{\mathbb{R}} \left(\mathrm{D}_x^{\frac{\alpha}{2}} \phi \right)^2 \, \mathrm{d}x + \frac{\alpha+1}{2} \int_{\mathbb{R}} \left(\mathrm{D}_x^{\frac{\alpha}{2}} \phi \right)^2 \, \mathrm{d}x.$$

Turning to the third statement, notice first that

$$\int_{\mathbb{R}} x^2 \left(\mathrm{D}_x^{\frac{\alpha}{2}} \phi \right)^2 \, \mathrm{d}x = -\langle (|\xi|^{\frac{\alpha}{2}} \xi \hat{\phi})_{\xi\xi}, |\xi|^{\frac{\alpha}{2}} \xi \overline{\hat{\phi}} \rangle$$
$$= -\left(\tfrac{\alpha}{2} + 1 \right) \tfrac{\alpha}{2} \int_{\mathbb{R}} \left(\mathrm{D}_x^{\frac{\alpha}{2}} \phi \right)^2 \, \mathrm{d}x + \langle \hat{\phi}_{\xi}, |\xi|^{\alpha} \xi^2 \overline{\hat{\phi}_{\xi}} \rangle, \tag{4.2}$$

where we used (4.1). Now,

$$\int_{\mathbb{R}} x^2 \phi_{xx} \mathrm{D}_x^{\alpha} \phi \, \mathrm{d}x = \langle (\xi^2 \hat{\phi})_{\xi\xi}, |\xi|^{\alpha} \hat{\phi} \rangle$$
$$= 2\langle \hat{\phi}, |\xi|^{\alpha} \overline{\hat{\phi}} \rangle + 4\langle \hat{\phi}_{\xi}, |\xi|^{\alpha} \xi \overline{\hat{\phi}} \rangle + \langle \hat{\phi}_{\xi\xi}, |\xi|^{\alpha} \xi^2 \overline{\hat{\phi}} \rangle$$
$$= 2\langle \hat{\phi}, |\xi|^{\alpha} \overline{\hat{\phi}} \rangle + (2-\alpha)\langle \hat{\phi}_{\xi}, |\xi|^{\alpha} \xi \overline{\hat{\phi}} \rangle - \langle \hat{\phi}_{\xi}, |\xi|^{\alpha} \xi^2 \overline{\hat{\phi}_{\xi}} \rangle$$
$$= -\int_{\mathbb{R}} x^2 \left(\mathrm{D}_x^{\frac{\alpha}{2}} \phi_x \right)^2 \, \mathrm{d}x + \tfrac{1}{4}(\alpha-2)^2 \int_{\mathbb{R}} \left(\mathrm{D}_x^{\frac{\alpha}{2}} \phi \right)^2 \, \mathrm{d}x,$$

by (4.1) and (4.2). □

Lemma 9 *(Properties of F) Let $\alpha > 0$, $G(\xi) = \mathrm{i}x\xi - |y|\xi(1+\xi^{\alpha})^{\frac{1}{2}}$ for $\xi \geq 0$ and $y \neq 0$. The function*

$$F(\xi) = \frac{\mathrm{d}}{\mathrm{d}\xi} \left(\frac{\xi}{(1+\xi^{\alpha})^{\frac{1}{2}}} \frac{1}{G'(\xi)} \right) \frac{1}{G'(\xi)} \quad \textit{for} \quad \xi \geq 0$$

satisfies

(a) $F(0) = \frac{1}{[G'(0)]^2} = (\mathrm{i}x - |y|)^{-2}$

(b) $|F'(\xi)| \lesssim T(\xi) \frac{1}{x^2+y^2}$*, for some function T such that $T e^{G} \in L^1(\mathbb{R}_+)$.*

Proof Let us first summarize all needed derivatives for the function G:

$$G(\xi) = \mathrm{i}x\xi - |y|\xi(1+\xi^{\alpha})^{\frac{1}{2}}$$
$$G'(\xi) = \mathrm{i}x - |y| \left((1+\xi^{\alpha})^{\frac{1}{2}} + \frac{\alpha}{2}\xi^{\alpha}(1+\xi^{\alpha})^{-\frac{1}{2}} \right)$$
$$= \frac{1}{2}(1+\xi^{\alpha})^{-\frac{1}{2}} \left(2\mathrm{i}x(1+\xi^{\alpha})^{\frac{1}{2}} - |y| \left(2 + (2+\alpha)\xi^{\alpha} \right) \right)$$
$$G''(\xi) = -\frac{\alpha}{4}|y|(1+\xi^{\alpha})^{-\frac{3}{2}}\xi^{\alpha-1} \left(2(1+\alpha) + (2+\alpha)\xi^{\alpha} \right)$$
$$G'''(\xi) = \frac{\alpha}{8}|y|(1+\xi^{\alpha})^{-\frac{5}{2}}\xi^{\alpha-2} \left(4(1+\xi^{\alpha})^2 - \alpha^2(\xi^{2\alpha} + 2\xi^{\alpha} + 4) \right)$$

Then, we compute F as

$$F(\xi) = \frac{1}{(1+\xi^\alpha)^{\frac{1}{2}}[G'(\xi)]^2} - \frac{\xi\left(\frac{\alpha}{2}(1+\xi^\alpha)^{-\frac{1}{2}}\xi^{\alpha-1}G'(\xi) + (1+\xi^\alpha)^{\frac{1}{2}}G''(\xi)\right)}{(1+\xi^\alpha)\,[G'(\xi)]^3},$$

which yields $F(0) = \frac{1}{[G'(0)]^2} = (\mathrm{i}x - |y|)^{-2}$ and proves part (a). A tedious, but straightforward computation yields that the derivative F is given by

$$\begin{aligned} F'(\xi) = &-\frac{\alpha}{2}\frac{(1+\alpha)\xi^{\alpha-1} + (1-\frac{\alpha}{2})\xi^{2\alpha-1}}{(1+\xi^\alpha)^{\frac{5}{2}}G'(\xi)^2} \\ &+ \frac{(\frac{3}{2}\alpha\xi^\alpha - 3(1+\xi^\alpha))G''(\xi) - (1+\xi^\alpha)\xi G'''(\xi)}{(1+\xi^\alpha)^{\frac{3}{2}}G'(\xi)^3} + \frac{3\xi G''(\xi)^2}{(1+\xi^\alpha)^{\frac{1}{2}}G'(\xi)^4} \\ =&: T_1(\xi) + T_2(\xi) + T_3(\xi). \end{aligned}$$

Now, we insert the expressions for G', G'', and G'''. We have

$$\begin{aligned} |G'(\xi)|^2 &= \frac{1}{4}(1+\xi^\alpha)^{-1}\left(4x^2(1+\xi^\alpha) + y^2(2+(2+\alpha)\xi^\alpha)^2\right) \gtrsim x^2 + y^2, \\ |G''(\xi)| &\eqsim |y|(1+\xi^\alpha)^{-\frac{1}{2}}\xi^{\alpha-1}, \\ |G'''(\xi)| &\eqsim |y|(1+\xi^\alpha)^{-\frac{5}{2}}\xi^{\alpha-2}\left|(4-\alpha^2)(1+\xi^\alpha)^2 - 3\alpha^2\right|. \end{aligned}$$

Starting with T_1 we estimate

$$|T_1(\xi)| \lesssim \frac{\xi^{\alpha-1}}{(1+\xi^\alpha)^{\frac{3}{2}}(x^2+y^2)}. \tag{4.3}$$

For T_2 we find

$$|T_2(\xi)| \lesssim |y|\frac{\xi^{\alpha-1}}{(1+\xi^\alpha)(x^2+y^2)^{\frac{3}{2}}}. \tag{4.4}$$

Eventually, we estimate T_3 as

$$|T_3(\xi)| \lesssim y^2\frac{\xi^{2\alpha-1}}{(1+\xi^\alpha)^{\frac{3}{2}}(x^2+y^2)^2}. \tag{4.5}$$

Summarizing (4.3)-(4.5), we find that $|F'(\xi)| \lesssim T(\xi)\frac{1}{x^2+y^2}$, where $Te^G \in L^1(\mathbb{R}_+)$ which proves part (b).

□

Lemma 10 *(Properties of E) Let* $\alpha > 0$, $G(\xi) = \mathrm{i}x\xi - |y|\xi(1+\xi^\alpha)^{\frac{1}{2}}$ *for* $\xi \geq 0$ *and* $y \neq 0$. *The function*

$$E(\xi) = \frac{1}{(1+\xi^\alpha)}\frac{1}{G'(\xi)} \quad \textit{for} \quad \xi \geq 0$$

satisfies

(a) $E(0) = \frac{1}{G'(0)} = \mathrm{i}x - |y|$

(b) $|E'(\xi)| \lesssim S(\xi)\frac{1}{x^2+y^2}$, *for some function S such that* $Se^G \in L^1(\mathbb{R}_+)$.

Proof The proof follows by direct computation. Recall that

$$|G'(\xi)|^2 = \frac{1}{4}(1+\xi^\alpha)^{-1}\left(4x^2(1+\xi^\alpha) + y^2(2+(2+\alpha)\xi^\alpha)^2\right) \gtrsim x^2 + y^2 \quad (4.6)$$

Part (a) follows immediately from $G'(0) = \mathrm{i}x - |y|$. For part (b), we compute the derivative of E' and use (4.6) to estimate

$$E'(\xi) = \frac{\alpha}{4}\frac{\xi^{\alpha-1}}{(1+\xi^\alpha)^2[G'(\xi)]^2} \lesssim \frac{\xi^{\alpha-1}}{(1+\xi^\alpha)^2}\frac{1}{x^2+y^2},$$

which yields the statement. □

Acknowledgements The authors would like to thank Christian Klein and Dmitry E. Pelinovsky for their helpful discussions. This research was carried out while D.N. was supported by the Wallenberg foundation.

Funding Open access funding provided by Lund University. Funding was provided by Knut och Alice Wallenbergs Stiftelse.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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