Piecewise-Tunneled Captive Processes and Corridored Random Particle Systems

Levent Ali Mengütürk[1] · **Murat Cahit Mengütürk[2]**

Abstract

We introduce a family of processes that generalises captive diffusions, whereby the stochastic evolution that remains within a pair of time-dependent boundaries can further be piecewise-tunneled internally. The tunneling effect on the dynamics can be random such that the process has non-zero probability to find itself within any possible tunnel at any given time. We study some properties of these processes and apply them in modelling corridored random particles that can be observed in fluid dynamics and channeled systems. We construct and simulate mean-reverting piecewise-tunneled captive models for demonstration. We also propose a doubly-stochastic system in which the tunnels themselves are generated randomly by another stochastic process that jumps at random times.

Keywords Captive diffusions · Particle systems · Bounded processes

Mathematics Subject Classification: 60

1 Introduction

Stochastic phenomenon restricted to evolve within given boundaries are commonly observed in numerous scientific fields including physics, engineering, biology and finance. It is thus by no surprise to find a fairly large body of academic literature involving random processes that are not allowed to leave some given domain. To name a few amongst many, we refer to reflected diffusions, reflected Ornstein–Uhlenbeck processes, Skorokhod-type stochastic differential equations (SDEs), diffusions on submanifolds, Bessel processes, Brownian excursions and non-colliding diffusions; see [1–18].

Communicated by Eric A. Carlen.

✉ Levent Ali Mengütürk
  ucaheng@ucl.ac.uk

  Murat Cahit Mengütürk
  murat.menguturk@ozyegin.edu.tr

[1] Department of Mathematics, University College London, London, UK

[2] Center for Financial Engineering, Ozyegin University, Istanbul, Turkey

For this work, the most relevant class of processes that cannot break free from exogenous boundaries are the *captive diffusions* of [19]—which are degenerate Markov processes, and are solutions to a family of SDEs with continuous coefficients satisfying some regularity conditions with respect to a pair of càdlàg paths that are right-differentiable where they are continuous. In [19], these processes are applied to model order-preserving dynamics, which may appear in different forms across different areas such as random matrix theory and interacting particle systems [20–22]. The aforementioned captive diffusions have later been used to construct Hermitian-valued diffusions on a convex cone of positive semi-definite matrices that evolve 'between' Hermitian-valued boundaries in the Loewner sense, which were used to solve a particular class of quadratic optimization problems that can generate captive efficient frontiers for purposes of volatility hedging and other risk management models (see, [23]).

In this paper, we are able to produce a model which not only encapsulates stochastic behaviour restricted to evolve within a given pair of boundaries, but can also admit randomly tunneled (or corridored) movements internally within the same restricted domain. Our motivation arises from channeling dynamics observed in physics [24–28], and from fluid dynamics where particles display random spatial variations [29–33]. Here, we aim to propose a tractable probabilistic framework that can be used to represent trajectories of random paths that evolve within multiple corridors with varying geometries. As such, we significantly generalise the captive diffusions of [19] using *path-dependent* SDE coefficients in relation to the given pair of boundaries and internal tunnel transitions. This paves way to a much wider range of stochastic behaviour under different state-spaces subject to multiple regime switches, translocational splits or bifurfactions. For example, our framework can be applied to modelling transitions between laminar flows and turbulent flows in a fluid system as the domain gets branched out into different flow regimes; another example of this behaviour can also be found in hemodynamics, where the varying geometry of the vessels directly governs the nature of blood flow. The framework may also find use in applications in high-throughput cytometry, intermolecular interactions during channel transports and bioparticle separation.

One of the main objectives of this paper is to investigate the little-researched phenomenon of how a stochastic particle adopts, and responds to, the complex geometry of a *maze-like* domain consisting of multiple corridors that also act, in themselves, as internal boundaries in a restricted state-space. In this sense, our framework considerably loosens the limitations of single-regime systems studied in the existing literature, such that the underlying geometry of the bounded state-space, in which a particle evolves, may lose its uniformity if divided into sub-spaces. This brings to mind several physical occurrences, like the motions of diffusing particles that are fired off into a multi-layered collider. In such a system, one can study the dynamics of these particles while they randomly enter and exit bifurcating tunnels (that may represent different regimes or force fields), and interact with interior walls at random, potentially gaining or losing energy as a result, and sometimes colliding with other particles in the system. If the experimenter seeks a purely controlled environment, the underlying principles of the model may help them design a pipeline where two particles are fired off simultaneously into two separate tunnels, such that they move in isolation for a pre-determined duration, and are later allowed to discharge into a wider (but still a bounded) domain where they collide with non-zero probability. The general environment of the two (or more) separate tunnels could be kept under dissimilar conditions at all times (e.g. by varying the elasticity of their interior walls, or the viscosity of their natural mediums, or the temperature of their surroundings) such that the particles, in their own right, could be subject to diverse habitats before being forced into a more unified state-space that allows particle interaction, while still maintaining the continuity of the underlying system. Then

again, the opposite scenario may also hold, such that two particles would commence their journey in chorus in a unified domain, only to be channeled into different environments defined by lateral tunnels, and later to be thrown into a new shared domain with exogenous boundaries, if needed—we will be able to model such scenarios as part of our application later in the paper. These questions may also become relevant when dealing with inertial lift forces in laminar microfluidic systems with multiple streamlines (see [34] for applications in a single-streamline scenario with randomly distributed particles). Considering the flexibility of our proposed architecture, examples of such nature may be further enriched, but are, by no means, exhaustive. For instance, our framework can be put in service to study inertial self-ordering and particle focusing in microchannels with symmetric or asymmetric geometries, or to generate regime-driven efficient frontiers in intertemporal risk minimisation problems, or to price more complex barrier options in finance—these are the research directions we leave for future. Furthermore, our framework allows one to control the Lebesgue measures of reflection at boundaries (see [2]) purely through the drift coefficients (a useful trait for calibration purposes in practice), which enable instantaneous reflections, delayed reflections or even absorptions at hitting times, depending on the nature of the underlying experiment. In essence, we aim to introduce a general mathematical recipe to construct a large family of stochastic processes that can be used to model, simulate and calibrate particle systems in restrictive geometries that admit a wide range of complexities through dynamic combinations of spatial fragmentation.

We keep this paper more on the applied side and leave a purely theoretical study for future. The structure of this work is as follows. In Sect. 2, we define what we call *piecewise-tunneled captive processes* as solutions to a family of path-dependent SDEs and study some of their properties. In Sect. 3, we construct explicit examples with mean-reverting behaviour to model corridored random particle systems and provide simulations for demonstration. In Sect. 4, we generalise the setup to the multivariate case, which allows one to introduce coupling effects between captive particles. We also produce a doubly-path-dependent extension in order to embed additional path memory into the evolution of particles. Finally, we propose a doubly-stochastic framework where tunnel trajectories are generated by a stochastic counting (jump) process that produces the internal random corridors. Section 5 concludes.

2 Piecewise-Tunneled Captive Processes

We shall start this section with a physical system to motivate the general mathematical setup we introduce later on. Imagine a particle that moves in time within a closed tube, never allowed to penetrate through its walls. The behaviour of the particle is further subject to forces that make its trajectories random within the tube. Now imagine that the tube gets more granular with the appearance of multiple internal corridors (or tunnels) at different times, which makes the movement of the particle more restricted whenever it finds itself in any of these corridors that have their own impenetrable walls. Assume further that the possible trajectories of the particle are affected by its current position and its distance to the walls (see, for example [35–37]), such that the particle gets stuck or absorbed in these walls if it hits any one of them.

As an example, we can model such dynamics using Brownian motion which is scaled by a polynomial coefficient as a function of the current position of the particle, where the map has its roots at the location of the walls. The following toy model of a particle $\{X_t\}_{0 \leq t < \infty}$

moving in time t captures the aforementioned dynamics:

$$X_t = x_0 + \int_0^t (X_s - L)(M - X_s + f_s)(U - X_s) \mathrm{d}W_s, \tag{1}$$

where $x_0 \in [L, U]$ is the initial point, $\{W_t\}_{0 \le t < \infty}$ is a Brownian motion, L and U form the constant positions of the lowermost and uppermost walls of the tube, and M is the position of an internal wall that divides the main tube into two narrower internal corridors, such that $L < M < U$. If we choose $f : \mathbb{R}_+ \to \mathbb{R}$ as a continuous function where $f_t \ne 0$ for $t < \bar{t}$ and $f_t = 0$ for $t \ge \bar{t}$ for some fixed time point $\bar{t} > 0$, then we have a model of a particle that satisfies the following:

1. $\{X_t\}_{0 \le t < \infty}$ always remains between L and U
2. If $\{X_t\}_{0 \le t < \infty}$ hits L or U at any time, it gets stuck there for the rest of its lifetime
3. The wall of the internal corridor M appears at time $\bar{t}$ and $\{X_t\}_{\bar{t} \le t < \infty}$ remains either within $[M, U]$ or within $[L, M]$ depending on which corridor the particle entered at time $\bar{t}$
4. Once in one of the internal corridors $[M, U]$ or $[L, M]$, if $\{X_t\}_{\bar{t} \le t < \infty}$ hits M, it gets stuck there for the rest if its lifetime

The equation above is a very specific example that we can propose. As such, we shall produce a significantly more general family of processes that can capture more complicated dynamics, while still being enclosed by outermost walls as well as internal tunnels whenever they appear. To the best of our knowledge, there is no work that can address the encapsulation of stochastic paths within heterogeneously-formed internal corridors in the generalisation we propose in this paper, which we manage to do so through path-dependent SDEs—for example, captive diffusions of [19] cannot capture internal corridors. This paper can essentially serve as a mathematical platform to construct bounded stochastic models within such geometries that can embed absorbed and reflected particles.

2.1 Main Results

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \le \infty}, \mathbb{P})$ be a filtered probability space, where all filtrations are right-continuous and complete, and where $\mathcal{F}_\infty = \mathcal{F}$. We use $\mathcal{B}$ to denote the Borel σ-field and choose a fixed time interval $\mathbb{T} = [0, T]$ for some $T < \infty$. We work on the Skorokhod space of càdlàg (right-continuous with left-limits) functions $\mathcal{D}(\mathbb{T} \times \mathbb{R}) \subset \Omega$, where $\mathcal{C}(\mathbb{T} \times \mathbb{R}) \subset \mathcal{D}(\mathbb{T} \times \mathbb{R})$ is its subspace of continuous paths. We use the map $X : \mathcal{C}(\mathbb{T} \times \mathbb{R}) \to \mathbb{R}$ to construct a continuous-time process $\{X_t\}_{t \in \mathbb{T}}$ and denote $\{\mathcal{F}_t^X\}_{t \in \mathbb{T}}$ as its natural filtration given by $\mathcal{F}_t^X = \sigma(\{X_s\} : 0 \le s \le t)$, such that $\mathcal{F}_t^X \subset \mathcal{F}_t$ is a sub-algebra for every $t \in \mathbb{T}$. We let $\mathcal{M}(\mathbb{T} \times \mathbb{R}) \subset \mathcal{C}(\mathbb{T} \times \mathbb{R})$ be the space of continuous $\mathbb{R}$-valued $(\mathbb{P}, \{\mathcal{F}_t\})$-martingales. We sometimes write $X(t) = X_t$ for $t \in \mathbb{T}$ interchangeably. We shall now define a family of functions which will be used throughout the paper to model time-dependent boundaries.

Definition 2.1 Let $\tilde{\mathcal{G}} \subset \mathcal{D}(\mathbb{T} \times \mathbb{R})$ be a set of measurable càdlàg functions, such that for any function $g \in \tilde{\mathcal{G}}$, $g : \mathbb{T} \to \mathbb{R}$ is a locally bounded map with a locally bounded right-derivative $\mathrm{d}g_+(t)/dt$ on the time-intervals where $g(t)$ is continuous. In addition, let $\mathcal{G} \subset \tilde{\mathcal{G}}$ be the subset of purely continuous functions that are locally bounded with locally bounded right-derivatives.

For any $g \in \tilde{\mathcal{G}}$, we denote $\Delta g_t = g_t - g_{t-}$ for all $t \in \mathbb{T}$, which means that $\Delta g_t = 0$ implies g is continuous at t and $\Delta g_t \ne 0$ implies there is discontinuity at t.

Definition 2.2 Let $\tilde{\mathcal{G}}^{(l)} \subset \tilde{\mathcal{G}}$ and $\tilde{\mathcal{G}}^{(u)} \subset \tilde{\mathcal{G}}$ be the subspaces such that

1. Every $g \in \tilde{\mathcal{G}}^{(l)}$ satisfies $\Delta g_t \leq 0$ for all $t \in \mathbb{T}$
2. Every $g \in \tilde{\mathcal{G}}^{(u)}$ satisfies $\Delta g_t \geq 0$ for all $t \in \mathbb{T}$

We note that the intersection of $\tilde{\mathcal{G}}^{(l)}$ and $\tilde{\mathcal{G}}^{(u)}$ is given by

$$\tilde{\mathcal{G}}^{(l)} \cap \tilde{\mathcal{G}}^{(u)} = \mathcal{G}.$$

In order to model internal corridors that can appear at different times, we shall work with different time-segments denoted by $\mathbb{T}^{(j)} \subseteq \mathbb{T}$ for $j = 0 \ldots, m \in \mathbb{N}_+$, for some fixed $1 \leq m < \infty$, where we define these segments as follows:

$$\mathbb{T}^{(j)} = [\tau_{\text{start}}^{(j)}, \tau_{\text{end}}^{(j)}] \text{with} 0 \leq \tau_{\text{start}}^{(j)} < \tau_{\text{end}}^{(j)} \leq T,$$

for $j = 0, \ldots, m$. For $j = 0$ and $j = m$, we specifically set

$$\tau_{\text{start}}^{(0)} = \tau_{\text{start}}^{(m)} = 0 \text{and} \tau_{\text{end}}^{(0)} = \tau_{\text{end}}^{(m)} = T,$$

which implies $\mathbb{T}^{(0)} = \mathbb{T}^{(m)} = \mathbb{T}$. We associate a boundary process $\{g_t^{(j)}\}_{t \in \mathbb{T}^{(j)}}$ to each of these time-segments, where $g^{(0)} \in \tilde{\mathcal{G}}^{(l)}$, $g^{(m)} \in \tilde{\mathcal{G}}^{(u)}$ and $g^{(j)} \in \mathcal{G}$ for any $j \neq 0$ and $j \neq m$ when $m > 1$. We order these boundaries such that

$$g_t^{(j)} < g_t^{(k)} \text{for any} j < k \text{at any} t \in \mathbb{T}^{(j)} \cap \mathbb{T}^{(k)} \neq \emptyset.$$

For $g_t^{(j)}$ and $g_t^{(k)}$ where $\mathbb{T}^{(j)} \cap \mathbb{T}^{(k)} = \emptyset$, the indexation amongst them can be pairwise-arbitrary as long as the remaining ordering scheme is preserved. Finally, we write

$$\mathbf{g}_t = \{g_t^{(0)}, \ldots, g_t^{(m)}\}$$

to collate all the boundaries, where if for a given $t \in \mathbb{T}$ there are $g_t^{(j)}$s not defined over that t, it should be understood that $\mathbf{g}_t$ does not include those $g_t^{(j)}$s at that $t \in \mathbb{T}$.

Remark 2.3 We refer to $g^{(0)}$ and $g^{(m)}$ as the *master boundaries*, since they will define the lowest and largest boundaries of our captive processes, respectively, running over the entire $\mathbb{T}$. When $m > 1$, we refer to $g^{(j)}$ for $j \neq 0$ and $j \neq m$ as the *tunneling (internal) boundaries*, since they remain between $g^{(0)}$ and $g^{(m)}$ over their respective time-segments.

We introduce a non-anticipative monitoring process $\{\Psi_t\}_{t \in \mathbb{T}}$ that keeps track of the values of our captive process $\{X_t\}_{t \in \mathbb{T}}$ (that we formally introduce in Definition 2.6 below) at the start of each time-segment. This will allow us to construct path-dependent coefficients that will ensure the captivity of our process within the master boundaries as well as the internal tunnels.

Definition 2.4 Let $\{\Psi_t\}_{t \in \mathbb{T}}$ be a set-valued process that records the values of $\{X_t\}_{t \in \mathbb{T}}$ at discrete time points $\tau_{\text{start}}^{(j)}$ at the start of each time-segment $\mathbb{T}^{(j)}$, given by

$$\Psi_t = \left\{ X_{\tau_{\text{start}}^{(j)}} : \tau_{\text{start}}^{(j)} \leq t, \text{ for } j = 0, \ldots, m \right\}. \tag{2}$$

Let $\{|\Psi_t|\}_{t \in \mathbb{T}}$ be the integer-valued process, where $|\Psi_t|$ is the number of elements of Ψ_t at $t \in \mathbb{T}$.

We shall clarify the notion of measurability for set-valued random variables since Ψ_t is a set-valued random variable for each $t \in \mathbb{T}$. First of all, with $m < \infty$, Ψ_t is a compact-set for every $t \in \mathbb{T}$. Hence, if $\mathcal{X}(\mathbb{R})$ denotes the family of compact subsets of $\mathbb{R}$, then

$$\Psi : \Omega \times \mathbb{T} \to \mathcal{X}(\mathbb{R})$$

is our random compact set, for which the measurability condition is given by

$$\{(\omega, t) \in \Omega \times \mathbb{T} : \Psi_t(\omega) \cap A \neq \emptyset\} \in \mathcal{F} \forall A \in \mathcal{B}(\mathbb{R}), \Psi_t(\omega) \in \mathcal{X}(\mathbb{R}). \tag{3}$$

Note that the measurability in (3) is similar to the definition of measurable functions, but where a set intersection modification appears to account for set-valued maps. We refer to [38–42] for definitions of measurability for set-valued random variables. For $\{\Psi_t\}_{t \in \mathbb{T}}$ to be adapted, we ask it to be progressively measurable, which means the map Ψ

$$(\omega, t) \mapsto \Psi_t(\omega) \text{is} \left(\mathcal{F}_t^X \otimes \mathcal{B}(\mathbb{T})\right) \text{-measurable} \forall t \in \mathbb{T}, \tag{4}$$

in the above sense. Finally, when we mean that a function is continuous with respect to Ψ, the topology we refer to is the continuity with respect to each element of Ψ_t at that $t \in \mathbb{T}$.

Remark 2.5 Note that $\{|\Psi_t|\}_{t \in \mathbb{T}}$ is a non-decreasing process, where $|\Psi_0| \geq 1$, since $\tau_{\text{start}}^{(0)} = \tau_{\text{start}}^{(m)} = 0$, and $|\Psi_T| \leq m$. That is, $\Psi_0 = \{X_0\}$ and we either have $\Psi_T = \{X_0\}$ if $m = 1$ or

$$\Psi_T = \{X_0, \ldots, X_{\tau_{\text{start}}^{(m-1)}}\},$$

if $m > 1$ such that if there are any repeating $\tau_{\text{start}}^{(j)}$ they collapse to one point.

We shall now introduce our main object; a class of stochastic processes that generalise the captive processes of [19]. For notational parsimony, we remain with $\mathbb{R}$-valued processes, and provide an $\mathbb{R}^n$-valued extension for $n \in \mathbb{N}_+$ later in the paper. We highlight that there is a deep and well-established literature on existence and uniqueness results for solutions of SDEs (see, [43, 44]). Since the main focus of this paper is not on existence and uniqueness, we do not specify particular sufficiency conditions (e.g. local Lipschitz continuity and linear growth) on the family of SDEs we work with, but instead, *define* our captive processes as solutions when they exist and are well-posed. We shall however prove an existence result later for an example model we will provide.

Definition 2.6 Keep the boundary setup as above. Then, a *piecewise-tunneled captive process* $\{X_t\}_{t \in \mathbb{T}} \in \mathcal{C}(\mathbb{T} \times \mathbb{R})$ is the solution to an SDE governed by

$$X_t = x_0 + \int_0^t \mu\left(s, \Psi_s, X_s; \mathbf{g}_s\right) ds + \int_0^t \sigma\left(s, \Psi_s, X_s; \mathbf{g}_s\right) dM_s, \tag{5}$$

with initial condition $X_0 = x_0 \in [g_0^{(0)}, g_0^{(m)})$, where μ and σ are continuous maps, satisfying the following conditions:

1. $\mu\left(t, \Psi_t, g_{t-}^{(j)}; \mathbf{g}_t\right) \geq dg_+^{(j)}(t)/dt + \Delta g_t^{(j)}$ if $X_{\tau_{\text{start}}^{(j)}} \geq g_{\tau_{\text{start}}^{(j)}}^{(j)}$, for any $t \in \mathbb{T}^{(j)}$ where $X_{t-} = g_{t-}^{(j)}$

2. $\mu\left(t, \Psi_t, g_{t-}^{(j)}; \mathbf{g}_t\right) \leq dg_+^{(j)}(t)/dt + \Delta g_t^{(j)}$ if $X_{\tau_{\text{start}}^{(j)}} < g_{\tau_{\text{start}}^{(j)}}^{(j)}$, for any $t \in \mathbb{T}^{(j)}$ where $X_{t-} = g_{t-}^{(j)}$

3. $\sigma\left(t, \Psi_t, g_{t-}^{(j)}; \mathbf{g}_t\right) = 0$, for any $t \in \mathbb{T}^{(j)}$ where $X_{t-} = g_{t-}^{(j)}$

for every $j = 0, \ldots, m$, where $\{M_t\}_{t \in \mathbb{T}} \in \mathcal{M}(\mathbb{T} \times \mathbb{R})$.

In Definition 2.6, $X(\tau_{\text{start}}^{(j)}) = g^{(j)}(\tau_{\text{start}}^{(j)})$ is associated to Property 1, which is a subjective choice, and may as well be associated to Property 2 if $x_0 \in (g_0^{(0)}, g_0^{(m)}]$ is the corresponding

initial condition. This is a purely technical point since we have $\mathbb{P}(X(\tau_{\text{start}}^{(j)}) = g^{(j)}(\tau_{\text{start}}^{(j)})) = 0$ for any $\tau_{\text{start}}^{(j)} > 0$ due to the continuity of $\{X_t\}_{t \in \mathbb{T}}$. The following statement is immediate from Definition 2.6.

Proposition 2.7 *Let* $\{W_t\}_{t \in \mathbb{T}}$ *be a* $(\mathbb{P}, \{\mathcal{F}_t^X\})$-*Brownian motion, and set* $M_t = W_t$ *for all* $t \in \mathbb{T}$ *in* (5). *Then a piecewise-tunneled captive process* $\{X_t\}_{t \in \mathbb{T}}$ *is a* $(\mathbb{P}, \{\mathcal{F}_t^X\})$-*Markov process if and only if*

$$\mu\left(t, \Psi_t, X_t; \boldsymbol{g}_t\right) = \mu\left(t, X_t; \boldsymbol{g}_t\right) \, and \, \sigma\left(t, \Psi_t, X_t; \boldsymbol{g}_t\right) = \sigma\left(t, X_t; \boldsymbol{g}_t\right).$$

Piecewise-tunneled captive processes boil down to the captive diffusions of [19] with $m = 1$ when Proposition 2.7 holds. Otherwise, Definition 2.6 extends [19] through path-dependent coefficients with respect to the monitoring process $\{\Psi_t\}_{t \in \mathbb{T}}$ that breaks the Markovian property, even if $M_t = W_t$ for all $t \in \mathbb{T}$. The possibility of having path-dependency is a key property for our result below.

Proposition 2.8 *The following hold* $\mathbb{P}$-*almost-surely:*

1. *If* $X_{\tau_{\text{start}}^{(j)}} \geq g_{\tau_{\text{start}}^{(j)}}^{(j)}$ *then* $X_t \geq g_t^{(j)}$ *for all* $t \in \mathbb{T}^{(j)}$,
2. *If* $X_{\tau_{\text{start}}^{(j)}} < g_{\tau_{\text{start}}^{(j)}}^{(j)}$ *then* $X_t \leq g_t^{(j)}$ *for all* $t \in \mathbb{T}^{(j)}/\{\tau_{\text{start}}^{(j)}\}$.

Proof We start with the case where $g^{(j)} \in \mathcal{G}$ for every $j = 1, \ldots, m$. Note that X_t is differentiable with respect to time if and only if $\sigma(t,) = 0$ at that $t \in \mathbb{T}$, given that μ is continuous, and hence, is locally bounded. Therefore, using property 3. in Definition 2.6, X_t is differentiable at every $X_t = g_t^{(j)}$. Now let $X(\tau_{\text{start}}^{(j)}) \geq g^{(j)}(\tau_{\text{start}}^{(j)})$. Since $g^{(j)}$ has locally bounded first right-derivatives and μ is locally bounded, using properties 1. and 3. in Definition 2.6, we have the following when $X_t = g_t^{(j)}$ for any $t \in \mathbb{T}^{(j)}$:

$$\mu\left(t, \Psi_t, g_t^{(j)}; \boldsymbol{g}_t\right) = \lim_{\epsilon \to 0^+} \frac{X_{t+\epsilon} - g_t^{(j)}}{\epsilon} \geq \lim_{\epsilon \to 0^+} \frac{g_{t+\epsilon}^{(j)} - g_t^{(j)}}{\epsilon},$$

and since $\{M_t\}_{t \in \mathbb{T}} \in \mathcal{M}(\mathbb{T} \times \mathbb{R})$ and μ and σ are continuous maps, $\{X_t\}_{t \in \mathbb{T}}$ has continuous paths $\mathbb{P}$-a.s., and applying mean-value theorem, $X_t \geq g_t^{(j)}$ must hold for all $t \in \mathbb{T}^{(j)}$. Now let $X(\tau_{\text{start}}^{(j)}) < g^{(j)}(\tau_{\text{start}}^{(j)})$. Then similarly, using properties 2. and 3. in Definition 2.6, we get the opposite side with $X_t = g_t^{(j)}$:

$$\mu\left(t, \Psi_t, g_t^{(j)}; \boldsymbol{g}_t\right) = \lim_{\epsilon \to 0^+} \frac{X_{t+\epsilon} - g_t^{(j)}}{\epsilon} \leq \lim_{\epsilon \to 0^+} \frac{g_{t+\epsilon}^{(j)} - g_t^{(j)}}{\epsilon},$$

for any $t \in \mathbb{T}^{(j)}/\{\tau_{\text{start}}^{(j)}\}$, which implies that $X_t \leq g_t^{(j)}$ for all $t \in \mathbb{T}^{(j)}/\{\tau_{\text{start}}^{(j)}\}$. Finally, when we allow $g^{(0)} \in \tilde{\mathcal{G}}^{(l)}$, $g^{(m)} \in \tilde{\mathcal{G}}^{(u)}$, we can further allow vertical adjustments to μ at the left-limits of the master boundaries, where the set of jumps

$$\mathcal{J} = \{t \in \mathbb{T} : \Delta g_t^{(0)} \neq 0 \cup \Delta g_t^{(m)} \neq 0\}$$

is countable and $X_{t-} = \lim_{s \to t, s < t} X_s = g^{(0)}(t-)$ implies $X_t = g^{(0)}(t-)$ and $X_{t-} = \lim_{s \to t, s < t} X_s = g^{(m)}(t-)$ implies $X_t = g^{(m)}(t-)$ $\mathbb{P}$-a.s., again due the continuity of $\{X_t\}_{t \in \mathbb{T}}$. We can thus follow similar steps as above with the additional adjustments of $\Delta g_t^{(0)} \leq 0$ and $\Delta g_t^{(m)} \geq 0$ at every $t \in \mathcal{J}$. $\square$

Proposition 2.8 tells us $\mathbb{P}$-a.s. that $\{X_t\}_{t\in\mathbb{T}}$ remains between the master boundaries

$$g_t^{(0)} \leq X_t \leq g_t^{(m)}$$

for all $t \in \mathbb{T}$, and within any given tunnel

$$g_t^{(j)} \leq X_t \leq g_t^{(k)}$$

with $j < k$ for all $t \in \mathbb{T}^{(j)} \cap \mathbb{T}^{(k)} \neq \emptyset$, if it *entered* that tunnel. This very behaviour gives the process $\{X_t\}_{t\in\mathbb{T}}$ its name in Definition 2.6. Before we give an example of a piecewise-tunneled captive process, we shall prove the following existence result that will be useful. Below, we denote $\mathcal{C}^1$ as the space of continuous functions that admit continuous first derivatives. We also define

$$\tau_t = 0 \vee \sup\{\tau_{\text{start}}^{(i)} : \tau_{\text{start}}^{(i)} \leq t, \forall i \in \mathcal{I}\},$$

where we adopt the convention $\sup \emptyset = -\infty$, and write $\mathcal{I} = \{1, \ldots, m-1\}$.

Proposition 2.9 *Choose $\{M_t\}_{t\in\mathbb{T}} = \{W_t\}_{t\in\mathbb{T}}$. Let μ and σ satisfy the conditions in Definition 2.6, and be $\mathcal{C}^1$ functions with respect to X and locally Lipschitz continuous functions with respect to the elements of Ψ for which we can write $\mu(\Psi_t) = \mu(X_0, \ldots, X_{\tau_t})$ and $\sigma(\Psi_t) = \sigma(X_0, \ldots, X_{\tau_t})$ for any $t \in \mathbb{T}$. Then solution (5) exists.*

Proof Since $\mu(\Psi_t) = \mu(X_0, \ldots, X_{\tau_t})$ and $\sigma(\Psi_t) = \sigma(X_0, \ldots, X_{\tau_t})$, local Lipschitz continuity of μ and σ with respect to the elements of Ψ can be understood in the usual sense – here all other parameters in the functions μ and σ are dropped for notational convenience. It remains to show that (5) satisfies local Lipschitz continuity given that μ and σ are $\mathcal{C}^1$ in X, which can be done in a standard way. First note that we have $x_0 \in [g_0^{(0)}, g_0^{(m)})$. Since $\mu : [g_0^{(0)}, g_0^{(m)}) \to \mathbb{R}$ and $\sigma : [g_0^{(0)}, g_0^{(m)}) \to \mathbb{R}$ are in $\mathcal{C}^1$, their derivatives μ' and σ' are continuous, and hence, locally bounded satisfying

$$|\mu'(x)| \leq K_\mu \text{ and } |\sigma'(x)| \leq K_\sigma$$

for all $x \in [g_0^{(0)}, g_0^{(m)})$, given that K_μ and K_σ are constants. Thus, for any $x_1, x_2 \in [g_0^{(0)}, g_0^{(m)})$ where $x_1 \leq x_2$, we have

$$|\mu(x_1) - \mu(x_2)| = |\mu'(y)||x_1 - x_2| \leq K_\mu|x_1 - x_2|$$
$$|\sigma(x_1) - \sigma(x_2)| = |\sigma'(y)||x_1 - x_2| \leq K_\sigma|x_1 - x_2|$$

by applying mean-value theorem, given that $y \in [g_0^{(0)}, g_0^{(m)})$. Thus, both μ and σ are locally Lipschitz continuous with respect to X which imply

$$|\mu(x_1) - \mu(x_2)| + |\sigma(x_1) - \sigma(x_2)| \leq (K_\mu + K_\sigma)|x_1 - x_2| \triangleq K|x_1 - x_2|. \tag{6}$$

Finally, since μ and σ are continuous (from Definition 2.6) and (6) holds, there exists a constant L such that the local linear growth condition is satisfied:

$$|\mu(x)| + |\sigma(x)| \leq L(1 + |x|).$$

It follows that (5) exists based on Itô existence theorem. $\qquad\square$

Example 2.10 Using Proposition 2.9, we can propose a piecewise-tunneled captive process with a dynamically mean-reverting drift coefficient and a multiplicative diffusion coefficient as follows:

$$X_t = x_0 + \int_0^t \kappa_s \left(\beta_s(\Psi_s; \mathbf{g}_s) - X_s \right) ds + \int_0^t \alpha_s \prod_{j=0}^m \left(f_s^{(j)}(g_s^{(j)}) - X_s \right) dW_s, \quad (7)$$

for $x_0 \in [g_0^{(0)}, g_0^{(m)})$, where $\{\alpha_t\}_{t \in \mathbb{T}}$ and $\{\kappa_t\}_{t \in \mathbb{T}} > 0$ are adapted continuous maps, $\{\beta_t\}_{t \in \mathbb{T}}$ is locally Lipschitz continuous with respect to the elements of Ψ where $\beta_t(\Psi_t) = \beta_t(X_0, \ldots, X_{\tau_t})$ for any $t \in \mathbb{T}$, and $\{f_t^{(j)}\}_{t \in \mathbb{T}}$ is an adapted continuous map where

$$\{f_t^{(j)}\}_{t \in \mathbb{T}^{(j)}} = \{g_t^{(j)}\}_{t \in \mathbb{T}^{(j)}}.$$

Here, σ satisfies property 2. in Definition 2.6. For μ to satisfy property 1. in Definition 2.6, $\{\beta_t\}_{t \in \mathbb{T}}$ needs to dynamically map the elements of Ψ into subdomains remaining between the tunnels – we shall provide an explicit example of (7) through mean-reversion dynamics later in the paper.

Note that μ in (7) is linear in X, and hence is $\mathcal{C}^1$ in X. In addition, σ in (7) is a polynomial function of X, and hence is also $\mathcal{C}^1$ in X – thus, solution (7) exists from Proposition 2.9.

Note that any piecewise-tunneled captive process $\{X_t\}_{t \in \mathbb{T}}$ is a $(\mathbb{P}, \{\mathcal{F}_t^X\})$-semimartingale given that μ has locally bounded variation. Then the following limits exist:

$$L_y(t)^+ = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_{\tau_{\text{start}}^{(j)}}^t \mathbf{1}\left(X_s \in [y, y+\epsilon] \right) d \langle X_s, X_s \rangle,$$

$$L_y(t)^- = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_{\tau_{\text{start}}^{(j)}}^t \mathbf{1}\left(X_s \in [y, y-\epsilon] \right) d \langle X_s, X_s \rangle,$$

for $t \in \mathbb{T}^{(j)}$ and some $\epsilon > 0$ and $y \in \mathbb{R}$, where $\mathbf{1}(A)$ denotes the indicator function for some $A \in \mathcal{B}(\mathbb{R})$, and $\langle ., . \rangle$ is the quadratic variation:

$$\langle X_t, X_t \rangle = \langle X_{\tau_{\text{start}}^{(j)}}, X_{\tau_{\text{start}}^{(j)}} \rangle + \int_{\tau_{\text{start}}^{(j)}}^t \sigma \left(s, \Psi_s, X_s; \mathbf{g}_s \right)^2 d \langle M_s, M_s \rangle,$$

and $L_y(t)^+$ and $L_y(t)^-$ are the right and left local times of $\{X_t\}_{t \in \mathbb{T}^{(j)}}$ at $y \in \mathbb{R}$, respectively. Note that local times can be used to quantify the amount of time $\{X_t\}_{t \in \mathbb{T}^{(j)}}$ spends at some given position. For the next statement, we let

$$\partial \mathcal{D}^{(j)} = \{ y : y \in \text{im}(g^{(j)}) \},$$

where $\text{im}(f)$ is the image of a function f.

Proposition 2.11 *Let* $\{X_t\}_{t \in \mathbb{T}}$ *be a* $(\mathbb{P}, \{\mathcal{F}_t^X\})$*-semimartingale and* $g^{(j)}$ *be constant such that* $g^{(j)}(t) = g^{(j)}(t')$ *for every* $t \neq t'$. *Then, for any* $y \in \partial \mathcal{D}^{(j)}$ *and* $t \in \mathbb{T}^{(j)}/\{\tau_{\text{start}}^{(j)}\}$ *for* $j = 1 \ldots, m$, *all of the following hold:*

1. $L_y(t)^+ = 2 \int_{\tau_{\text{start}}^{(j)}}^t \mathbf{1}\left(X_s = y \right) \mu \left(s, \Psi_s, X_s; \mathbf{g}_s \right) ds$ *if* $X_{\tau_{\text{start}}^{(j)}} \geq g_{\tau_{\text{start}}^{(j)}}^{(j)}$

2. $L_y(t)^+ = 0$ *if* $X_{\tau_{\text{start}}^{(j)}} < g_{\tau_{\text{start}}^{(j)}}^{(j)}$

3. $L_y(t)^- = -2 \int_{\tau_{\text{start}}^{(j)}}^t \mathbf{1}\left(X_s = y \right) \mu \left(s, \Psi_s, X_s; \mathbf{g}_s \right) ds$ *if* $X_{\tau_{\text{start}}^{(j)}} < g_{\tau_{\text{start}}^{(j)}}^{(j)}$

4. $L_y(t)^- = 0$ *if* $X_{\tau_{\text{start}}^{(j)}} \geq g_{\tau_{\text{start}}^{(j)}}^{(j)}$

Proof Define the functions $\text{sgn}(x)^+ = \mathbf{1}(x > 0) - \mathbf{1}(x \leq 0)$ and $\text{sgn}(x)^- = \mathbf{1}(x \geq 0) - \mathbf{1}(x < 0)$, and let $y \in \partial \mathcal{D}^{(j)}$. For $t \in \mathbb{T}^{(j)}$, using Tanaka formula, we have the following:

$$L_y(t)^+ = |X_t - y| - |X_{\tau_{\text{start}}^{(j)}} - y| - \int_{\tau_{\text{start}}^{(j)}}^t \text{sgn}(X_s - y)^+ \mathrm{d}X_s \tag{8}$$

$$L_y(t)^- = |X_t - y| - |X_{\tau_{\text{start}}^{(j)}} - y| - \int_{\tau_{\text{start}}^{(j)}}^t \text{sgn}(X_s - y)^- \mathrm{d}X_s. \tag{9}$$

From Proposition 2.8, if $X_{\tau_{\text{start}}^{(j)}} \geq g_{\tau_{\text{start}}^{(j)}}^{(j)}$, then $X_t \geq y$ $\mathbb{P}$-a.s. for any $t \in \mathbb{T}^{(j)}$, given that $g^{(j)} = y$ is constant. Hence, using (8), we get

$$\begin{aligned}
L_y(t)^+ &= X_t - X_{\tau_{\text{start}}^{(j)}} - \int_{\tau_{\text{start}}^{(j)}}^t \mathbf{1}(X_s > y)\, \mu\left(s, \Psi_s, X_s; \mathbf{g}_s\right) \mathrm{d}s \\
&\quad + \int_{\tau_{\text{start}}^{(j)}}^t \mathbf{1}(X_s \leq y)\, \mu\left(s, \Psi_s, X_s; \mathbf{g}_s\right) \mathrm{d}s \\
&\quad - \int_{\tau_{\text{start}}^{(j)}}^t \mathbf{1}(X_s > y)\, \sigma\left(s, \Psi_s, X_s; \mathbf{g}_s\right) \mathrm{d}M_s \\
&\quad + \int_{\tau_{\text{start}}^{(j)}}^t \mathbf{1}(X_s \leq y)\, \sigma\left(s, \Psi_s, X_s; \mathbf{g}_s\right) \mathrm{d}M_s \\
&= X_t - X_{\tau_{\text{start}}^{(j)}} + 2 \int_{\tau_{\text{start}}^{(j)}}^t \mathbf{1}(X_s = y)\, \mu\left(s, \Psi_s, X_s; \mathbf{g}_s\right) \mathrm{d}s \\
&\quad - \int_{\tau_{\text{start}}^{(j)}}^t \mu\left(s, \Psi_s, X_s; \mathbf{g}_s\right) \mathrm{d}s - \int_{\tau_{\text{start}}^{(j)}}^t \sigma\left(s, \Psi_s, X_s; \mathbf{g}_s\right) \mathrm{d}M_s \\
&= 2 \int_{\tau_{\text{start}}^{(j)}}^t \mathbf{1}(X_s = y)\, \mu\left(s, \Psi_s, X_s; \mathbf{g}_s\right) \mathrm{d}s.
\end{aligned}$$

From Proposition 2.8, if $X_{\tau_{\text{start}}^{(j)}} < g_{\tau_{\text{start}}^{(j)}}^{(j)}$, then $X_t \leq y$ $\mathbb{P}$-a.s. for any $t \in \mathbb{T}/\{\tau_{\text{start}}^{(j)}\}$, with $g^{(j)} = y$. Hence, we get

$$\begin{aligned}
L_y(t)^+ &= X_{\tau_{\text{start}}^{(j)}} - X_t + \int_{\tau_{\text{start}}^{(j)}}^t \mathbf{1}(X_s \leq y)\, \mu\left(s, \Psi_s, X_s; \mathbf{g}_s\right) \mathrm{d}s \\
&\quad + \int_{\tau_{\text{start}}^{(j)}}^t \mathbf{1}(X_s \leq y)\, \sigma\left(s, \Psi_s, X_s; \mathbf{g}_s\right) \mathrm{d}M_s \\
&= X_{\tau_{\text{start}}^{(j)}} - X_t + \int_{\tau_{\text{start}}^{(j)}}^t \mathrm{d}X_s = 0.
\end{aligned}$$

On the other hand, using (9), since $X_{\tau_{\text{start}}^{(j)}} < g_{\tau_{\text{start}}^{(j)}}^{(j)}$ implies $X_t \leq y$ $\mathbb{P}$-a.s. for any $t \in \mathbb{T}/\{\tau_{\text{start}}^{(j)}\}$ as above, we have the following:

$$\begin{aligned}
L_y(t)^- &= X_{\tau_{\text{start}}^{(j)}} - X_t - \int_{\tau_{\text{start}}^{(j)}}^t \mathbf{1}(X_s \geq y)\, \mu\left(s, \Psi_s, X_s; \mathbf{g}_s\right) \mathrm{d}s \\
&\quad + \int_{\tau_{\text{start}}^{(j)}}^t \mathbf{1}(X_s < y)\, \mu\left(s, \Psi_s, X_s; \mathbf{g}_s\right) \mathrm{d}s
\end{aligned}$$

$$- \int_{\tau_{\text{start}}^{(j)}}^{t} \mathbf{1}\left(X_s \geq y\right) \sigma\left(s, \Psi_s, X_s; \mathbf{g}_s\right) dM_s$$

$$+ \int_{\tau_{\text{start}}^{(j)}}^{t} \mathbf{1}\left(X_s < y\right) \sigma\left(s, \Psi_s, X_s; \mathbf{g}_s\right) dM_s$$

$$= X_{\tau_{\text{start}}^{(j)}} - X_t - 2 \int_{\tau_{\text{start}}^{(j)}}^{t} \mathbf{1}\left(X_s = y\right) \mu\left(s, \Psi_s, X_s; \mathbf{g}_s\right) ds$$

$$+ \int_{\tau_{\text{start}}^{(j)}}^{t} \mu\left(s, \Psi_s, X_s; \mathbf{g}_s\right) ds + \int_{\tau_{\text{start}}^{(j)}}^{t} \sigma\left(s, \Psi_s, X_s; \mathbf{g}_s\right) dM_s^{(i)}$$

$$= -2 \int_{\tau_{\text{start}}^{(j)}}^{t} \mathbf{1}\left(X_s = y\right) \mu\left(s, \Psi_s, X_s; \mathbf{g}_s\right) ds.$$

Following similar steps as above, we finally get $L_y(t)^- = 0$ if $X_{\tau_{\text{start}}^{(j)}} \geq g_{\tau_{\text{start}}^{(j)}}^{(j)}$. $\qquad\square$

Remark 2.12 Note that if we have

$$\mathbb{P}\left(\int_{\tau_{\text{start}}^{(j)}}^{t} \mathbf{1}\left(X_s = y\right) \mu\left(s, \Psi_s, X_s; \mathbf{g}_s\right) ds = 0\right) = 1,$$

then from Proposition 2.11, we have

$$L_y(t)^+ = L_y(t)^- = 0,$$

in any scenario for $t \in \mathbb{T}^{(j)}/\{\tau_{\text{start}}^{(j)}\}$, $j = 1 \dots, m$.

In certain cases, we would want to work with piecewise-tunneled captives which are strict $(\mathbb{P}, \{\mathcal{F}_t^X\})$-martingales. For this, we would need to have $g^{(j)}$s satisfy certain characteristics. As such, for our next result, we define the time-segments $\mathcal{T}^{(j)} \subseteq \mathbb{T}^{(j)}$ where the probability that $\{X_t\}_{t \in \mathbb{T}}$ may hit the corresponding boundary is larger than zero:

$$\mathcal{T}^{(j)} = \left\{t : \mathbb{P}\left(X_t \in \delta g^{(j)}(t)\right) > 0; \ \text{for} t \in \mathbb{T}^{(j)}\right\},$$

where $\delta g^{(j)}(t)$ is the spatial differential of $g^{(j)}(t)$ at a fixed time $t \in \mathbb{T}^{(j)}$.

Proposition 2.13 *If the piecewise-tunneled captive process $\{X_t\}_{t \in \mathbb{T}}$ is a $(\mathbb{P}, \{\mathcal{F}_t^X\})$ -martingale, then all of the following must hold:*

1. *$-\Delta g_t^{(0)} \geq dg_+^{(0)}(t)/dt$ for all $t \in \mathcal{T}^{(0)}$,*
2. *$-\Delta g_t^{(m)} \leq dg_+^{(m)}(t)/dt$ for all $t \in \mathcal{T}^{(m)}$,*
3. *Each $g_t^{(j)}$ for $j \neq 0$ and $j \neq m$ is constant for all $t \in \mathcal{T}^{(j)}$*

Proof If $\{X_t\}_{t \in \mathbb{T}}$ is a $(\mathbb{P}, \{\mathcal{F}_t^X\})$-martingale, then we must have $\mu\left(t, \Psi_t, X_t; \mathbf{g}_t\right) = 0$ for all $t \in \mathbb{T}$. Thus, from Definition 2.6, we must have

$$0 \geq dg_+^{(0)}(t)/dt + \Delta g_t^{(0)},$$

for all $t \in \mathcal{T}^{(0)}$ due to property 1, and the following:

$$0 \leq dg_+^{(m)}(t)/dt + \Delta g_t^{(m)},$$

for all $t \in \mathcal{T}^{(m)}$ due to property 2. For any other $j \neq 0$ and $j \neq m$ where $g^{(j)} \in \mathcal{G}$ so that $\Delta g_t^{(j)} = 0$, all possible trajectories of $\{X_t\}_{t \in \mathbb{T}}$ dictates that either of the following will apply:

$$X(\tau_{\text{start}}^{(j)}) \geq g^{(j)}(\tau_{\text{start}}^{(j)}) \text{ or } X(\tau_{\text{start}}^{(j)}) < g^{(j)}(\tau_{\text{start}}^{(j)}),$$

from all possible path realizations $\mathbb{P}$-a.s. Since both property 1 and property 2 in Definition 2.6 must hold, and having $\mu\left(t, \Psi_t, X_t; \mathbf{g}_t\right) = 0$ for all $t \in \mathbb{T}$, we must have

$$0 \leq dg_+^{(j)}(t)/dt \leq 0$$

which means $dg_+^{(j)}(t)/dt = 0$ for all $t \in \mathcal{T}^{(j)}$.   □

When the master boundaries are continuous everywhere, the statement below follows immediately from Proposition 2.13—that is, the master boundaries must be monotonic.

Remark 2.14 If all boundaries are continuous with $\Delta g_t^{(0)} = 0$ and $\Delta g_t^{(m)} = 0$ for all $t \in \mathbb{T}$ so that $g^{(0)}, g^{(m)} \in \mathcal{G}$, and $\{X_t\}_{t \in \mathbb{T}}$ is a $(\mathbb{P}, \{\mathcal{F}_t^X\})$-martingale, then

1. $g^{(0)}$ is non-increasing over $\mathcal{T}^{(0)}$
2. $g^{(m)}$ is non-decreasing over $\mathcal{T}^{(m)}$
3. Each $g^{(j)}$ for $j \neq 0$ and $j \neq m$ is constant over $\mathcal{T}^{(j)}$

If for the martingale case, *all* boundaries are constant, then the boundaries must be absorbing. That is, if $\{X_t\}_{t \in \mathbb{T}}$ hits any of the constant boundaries at any given time, then the process gets stuck there for the rest of its lifetime – recall the example we considered in (1), which is a martingale.

Proposition 2.15 *If $\{X_t\}_{t \in \mathbb{T}}$ is a $(\mathbb{P}, \{\mathcal{F}_t^X\})$-martingale and $g^{(j)}$ is constant such that $g^{(j)}(t) = g^{(j)}(t')$ for every $t \neq t'$, then $\{X_t\}_{t \in \mathbb{T}}$ gets absorbed in $g^{(j)}$ as of its first hitting-time.*

Proof Let the random variable $\rho^{(j)}$ be the first hitting-time to boundary $g^{(j)}$ given by

$$\rho^{(j)} = \inf\{t \geq 0 : X_t = g^{(j)}\},$$

where we adopt the convention $\inf \emptyset = \infty$. Define $\{Y_t^{(j)}\}_{t \in \mathbb{T}^{(j)}} = \{X_{t \wedge \rho^{(j)}}\}_{t \in \mathbb{T}^{(j)}}$ be a stopped piecewise-tunneled captive process. If $\{X_t\}_{t \in \mathbb{T}}$ is a $(\mathbb{P}, \{\mathcal{F}_t^X\})$-martingale and $g^{(j)}$ is constant, then

$$\{X_t\}_{t \in \mathbb{T}^{(j)}} = \{Y_t^{(j)}\}_{t \in \mathbb{T}^{(j)}},$$

must hold since $\mu\left(t, \Psi_t, X_t; \mathbf{g}_t\right) = 0$ for all $t \in \mathbb{T}$.   □

More generally, $\{X_t\}_{t \in \mathbb{T}}$ can demonstrate absorbing or reflecting dynamics with respect to its master boundaries and its internal tunnels. In fact, $\{X_t\}_{t \in \mathbb{T}}$ can display both absorbing and reflecting behaviour over non-overlapping time horizons depending on how the μ coefficient evolves over time.

Remark 2.16 It would be interesting to apply piecewise-tunneled captives within the field of stochastic partial differential equations (SPDEs), where the random noise component belongs to our proposed family of processes. We leave this direction for future research as it merits dedicated analysis.

3 Corridored Random Particle Systems

We shall apply piecewise-tunneled captives to model random particle systems that can arise in various fields ranging from fluid dynamics in microchannels to regime-switching population models under restricted conditions. The specific setup below, in principle, has certain overlaps with the literature on reflected diffusions in confined geometries. Reflected diffusion processes, in general, represent systems in which random paths exhibit continuous returns to the interiors of a single state-space after hitting an impassable boundary. The length of time that these processes spend at the boundary determines whether they are subject to an instantaneous reflection (with zero Lebesgue measure) or a delayed reflection (with positive Lebesgue measure), such that the underlying processes demonstrate, by virtue of their constitution, a set of varying characteristics of first-time passages. The physics of the phenomenon is sometimes linked to the movements of a particle whose velocity at the boundary is identical to the velocity of its exit. In a world of elastic boundaries, a particle with finite velocity would undergo a delayed reflection (with a partially-reduced velocity at the reflection point) to ensure finite displacement in a positive duration of time (see [2]).

The mean-reversion aspect of reflected diffusions have gained considerable attention in the literature. Spectral methods have been used in [13] to provide clear expressions for reflected Ornstein–Uhlenbeck processes (ROU) and the associated transition densities. Some of the underlying assumptions were applied to queuing systems, such as [10], which studied both the steady-state and the transient behaviour of the Ornstein–Uhlenbeck diffusions with a zero reflecting boundary, together with their moments and first-time passages when the corresponding boundary is significantly distanced from the origin. This model was later expanded by [15] which investigated the Laplace transform of the first-passage times of the Ornstein–Uhlenbeck diffusions with two-sided boundaries (see [45] for another application in queuing systems). Recently [46] discovered an interesting connection between Cox-Ingersoll-Ross process and ROU, arguing that, with probability 1, the square root of the former converges uniformly to the latter as the mean-reversion parameter tends to 0 in the fractional case. In finance, while [47] focused on interest rate models under reflecting and absorbing boundaries, [48] studied the pricing of barrier options on zero-coupon bonds. As an extension, [49] computed the conditional default probabilities of the price dynamics (e.g. digital options) driven by reflected SDEs with two-sided barriers, subject to a regulated market. Similar principles, concerning the evolution of stochastic diffusions in restrictive conditions, are applied to population dynamic models as well (see [50, 51]). Furthermore, the reader is guided to [52, 53] for other recent developments.

Related literature in biology deals with multi-cellular processes such that the microscopic particles move in and out of channels (e.g. pores or nanopores) via the mechanism called molecular translocation, where the molecule, pore and intermolecular interactions have direct influence on the current. Empirical evidence shows that the speed of molecules tend to decrease when they enter into the channels with entropic barriers. By defining channel transport as a chemical transition from one discrete state to another, [36] make use of discrete stochastic models to investigate intermolecular interactions and molecular translocation through nanopores, and find that particle currents display differing dynamics near the entrance and the exit of a channel (see further discussions in [54, 55]). We also refer the reader to recent studies in fluid mechanics dealing with near-wall dynamics and confined turbulent flows (see [37, 56–59]) to shed further light as to how particles may show variations in their behaviour as they draw closer to the walls of a confined geometry – an idea that overlaps with

our work through the natural observation of piecewise-tunneled captives displaying varying particle volatility as the diffusion approaches the exogenous walls of the tunnels.

In light of the existing literature that tends to focus on single state-spaces and usually a single class of mean-reverting processes, our framework extends the insightful works above by introducing the ability and flexibility of moulding the very topography of the underlying system by splitting the state-space into multiple internal tunnels (or regimes) of almost any length and shape, which enriches the domain with further complexity without forfeiting the continuity and tractability of the model; a setup which standard models cannot address. This gives the experimenter a useful platform to design sophisticated pipelines of varying constitutions that would directly govern the evolution of the random diffusions within. Moreover, by the generality in which the piecewise-tunneled captive diffusions are being defined, our framework admits a much wider class of stochastic processes currently studied in the related literature. This highlights the fact that the scope of piecewise-tunneled captives includes such stochastic processes that may or may not be mean-reverting, potentially adding further versatility to the design of experiments.

For our numerical setup, we denote $\{\hat{X}_{t_k}\}_{t_k \in \mathbb{T}}$ as a discretized approximation of $\{X_t\}_{t \in \mathbb{T}}$ over $0 = t_0 \leq t_1 \leq \ldots \leq t_p \leq = T < \infty$ for some $p \in \mathbb{N}_+$. For the simulations below, we use the Euler-Maruyama scheme

$$\hat{X}_{t_{k+1}} = \hat{X}_{t_k} + \mu\left(t_k, \hat{\Psi}_{t_k}, \hat{X}_{t_k}; \mathbf{g}_{t_k}\right)\delta + \sigma\left(t_k, \hat{\Psi}_{t_k}, \hat{X}_{t_k}; \mathbf{g}_{t_k}\right)\left(W_{t_{k+1}} - W_{t_k}\right),$$

where $\hat{X}_{t_0} = x_0$, $\delta = T/p$ and $t_k = k\delta$. Below, we choose $T = 1$ and $p = 10000$. We construct three examples that belong to the mean-reverting family in Example 2.10 for a parsimonious demonstration. Surely, the examples below are far from being exhaustive as one can increase the complexity of models significantly using Definition 2.6.

3.1 Single Inner-Boundary

Consider a particle system that evolves stochastically inside a tube. An additional internal boundary divides this tube into two smaller tunnels over a given (possibly shorter) time frame. Assume that the random evolutions of the particles within this domain demonstrate mean-reverting behaviour and the dynamics are influenced by the distance of particles to any given wall. The mean-reversion in turn causes particles to reflect back from the walls with which they collide. The SDE in (7) in Example 2.10 can model such a system, for which the solution exists from Proposition 2.9.

Accordingly, we set $m = 2$ and fix $\{\alpha_t\}_{t \in \mathbb{T}} = \alpha$, $\{\kappa_t\}_{t \in \mathbb{T}} = \kappa$ as constants for parsimony. The parameter α quantifies the strength of the diffusion coefficient on the particle dynamics and κ can be viewed as the strength of the mean-reversion. We also let the boundaries be constant such that $\{g_t^{(j)}\}_{t \in \mathbb{T}} = c^{(j)}$ where $-\infty < c^{(0)} < c^{(1)} < c^{(2)} < \infty$ holds. As for the time-segments, we already know that $\mathbb{T}^{(0)} = \mathbb{T}^{(m)} = \mathbb{T}$, so we need to specify $\mathbb{T}^{(1)}$ for the running-time of the internal boundary and choose some $0 \leq \tau_{\text{start}}^{(1)} < \tau_{\text{end}}^{(1)} \leq T$. The diffusion coefficient is given by

$$\sigma\left(t, \Psi_t, X_t, ; \mathbf{g}_s\right) = (c^{(0)} - X_t)(c^{(1)} - X_t + V_t)(c^{(2)} - X_t),$$

where $\{V_t\}_{t \in \mathbb{T}}$ is a bounded continuous map such that $V_t = 0$ for $t \in \mathbb{T}^{(1)}$ – e.g. we choose

$$V_t = r_1 * (\max(\tau_{\text{start}}^{(1)} - t, 0) - r_2 * \max(t - \tau_{\text{end}}^{(1)}), 0),$$

for $r_1, r_2 \in \{-1, 1\}$. For the examples below, we shall choose $r_1 = 1$ and $r_2 = -1$. It remains to model the drift coefficient μ, that controls the mean-reversion dynamics, given by

$$\mu\left(t, \Psi_t, X_t; \mathbf{g}_t\right) = \beta_t(\Psi_t; \mathbf{g}_t) - X_t,$$

where we can write $\beta_t(\Psi_t) = \beta_t(X(0), \ldots, X_{\tau_t})$ as in Proposition 2.9. More explicitly, we let $\{\beta_t\}_{t \in \mathbb{T}}$ take the form below:

$$\beta_t(\Psi_t; \mathbf{g}_t) = \begin{cases} c^{(1)} & \text{for } t \in [0, \tau_{\text{start}}^{(1)}), \\[2ex] c^{(1)} + \frac{c^{(2)} - c^{(1)}}{2(\tau_{\text{end}}^{(1)} - \tau_{\text{start}}^{(1)})}(t - \tau_{\text{start}}^{(1)}) & \text{for } t \in [\tau_{\text{start}}^{(1)}, \tau_{\text{end}}^{(1)}) \text{ if } X_{\tau_{\text{start}}^{(1)}} \geq c^{(1)}, \\[2ex] c^{(1)} - \frac{c^{(1)} - c^{(0)}}{2(\tau_{\text{end}}^{(1)} - \tau_{\text{start}}^{(1)})}(t - \tau_{\text{start}}^{(1)}) & \text{for } t \in [\tau_{\text{start}}^{(1)}, \tau_{\text{end}}^{(1)}) \text{ if } X_{\tau_{\text{start}}^{(1)}} < c^{(1)}, \\[2ex] \frac{c^{(1)} + c^{(2)}}{2} & \text{for } t \in [\tau_{\text{end}}^{(1)}, T] \quad \text{if } X_{\tau_{\text{start}}^{(1)}} \geq c^{(1)}, \\[2ex] \frac{c^{(0)} + c^{(1)}}{2} & \text{for } t \in [\tau_{\text{end}}^{(1)}, T] \quad \text{if } X_{\tau_{\text{start}}^{(1)}} < c^{(1)} \end{cases}$$

The function μ is bounded, continuous and satisfies properties 1. and 2. in Definition 2.6. Also, $\{\beta_t\}_{t \in \mathbb{T}}$ can be represented as the sum of locally Lipschitz continuous functions

$$\beta_t(\Psi_t; \mathbf{g}_t) = c^{(1)} \mathbf{1}\left(t \in [0, \tau_{\text{start}}^{(1)})\right)$$

$$+ \left(c^{(1)} + \frac{c^{(2)} - c^{(1)}}{2(\tau_{\text{end}}^{(1)} - \tau_{\text{start}}^{(1)})}(t - \tau_{\text{start}}^{(1)})\right)\mathbf{1}\left(t \in [\tau_{\text{start}}^{(1)}, \tau_{\text{end}}^{(1)}) \bigcap X_{\tau_{\text{start}}^{(1)}} \geq c^{(1)}\right)$$

$$+ \left(c^{(1)} - \frac{c^{(1)} - c^{(0)}}{2(\tau_{\text{end}}^{(1)} - \tau_{\text{start}}^{(1)})}(t - \tau_{\text{start}}^{(1)})\right)\mathbf{1}\left(t \in [\tau_{\text{start}}^{(1)}, \tau_{\text{end}}^{(1)}) \bigcap X_{\tau_{\text{start}}^{(1)}} < c^{(1)}\right)$$

$$+ \left(\frac{c^{(1)} + c^{(2)}}{2}\right)\mathbf{1}\left(t \in [\tau_{\text{end}}^{(1)}, T] \bigcap X_{\tau_{\text{start}}^{(1)}} \geq c^{(1)}\right)$$

$$+ \left(\frac{c^{(0)} + c^{(1)}}{2}\right)\mathbf{1}\left(t \in [\tau_{\text{end}}^{(1)}, T] \bigcap X_{\tau_{\text{start}}^{(1)}} < c^{(1)}\right),$$

where each element is associated to the indicator function $\mathbf{1}(.)$ as above, and hence, $\{\beta_t\}_{t \in \mathbb{T}}$ is locally Lipschitz continuous with respect to the elements of Ψ. We now have an explicit model for a piecewise-tunneled captive process with reflecting boundaries. Figure 1 below provides four sample paths of four particles that follow the aforementioned dynamics.

We see in Fig. 1 that whenever a particle enters one of the two internal tunnels, it remains in that tunnel until it leaves that tunnel to move more freely again, while still remaining within the master boundaries. For example, in the top-left panel, three of the particles find themselves within the upper corridor and the remaining one evolves within the lower corridor until the tunnels open up again. On the other hand, in the bottom-right panel, two particles evolve within the upper corridor and the other two evolve within the lower corridor.

In each panel, whenever any particle hits any of the walls, it is reflected back. We can surely move the time-location of the tunnels by changing $\tau_{\text{start}}^{(1)}$ and $\tau_{\text{end}}^{(1)}$, as shown below.

If the proposed captive process is used to model fluid flows within the given geometry above in Fig. 2, we can interpret the simulations such that on the left, we see a transition from
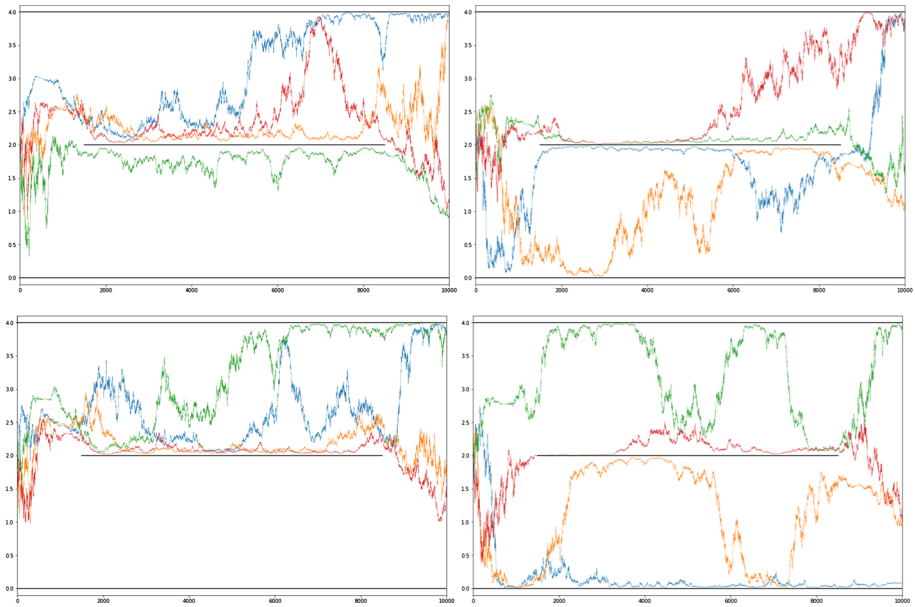
Fig. 1 Here, $\alpha = 1$, $\kappa = 2$, $c^{(0)} = 0$, $c^{(1)} = 2$, $c^{(2)} = 4$, $\tau_{\text{start}}^{(1)} = 1500$, $\tau_{\text{end}}^{(1)} = 8500$, $x_0 = 2$
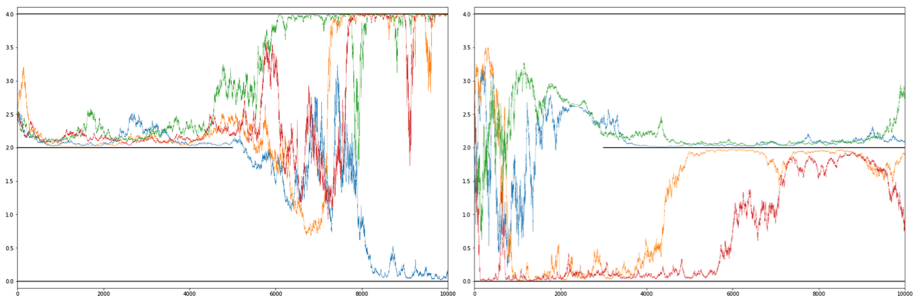

Fig. 2 Left: $\tau_{\text{start}}^{(1)} = 0$, $\tau_{\text{end}}^{(1)} = 5000$, $x_0 = 2.5$, Right: $\tau_{\text{start}}^{(1)} = 3000$, $\tau_{\text{end}}^{(1)} = 10000$, Right: $x_0 = 1$

a laminar flow to a turbulent flow as the tunnel opens up, and on the right, vice versa—here, the particles tend to behave in a more volatile manner when they evolve outside of the internal tunnels.

In fluid dynamics, we can use $\{\mu\left(t, \Psi_t, X_t; \mathbf{g}_t\right)\}_{t \in \mathbb{T}}$ to govern the average characteristic of the flow (e.g. velocity, viscosity) and $\{\sigma\left(t, \Psi_t, X_t; \mathbf{g}_t\right)\}_{t \in \mathbb{T}}$ the fluctuations, whereby the ratio μ/σ drives the prevailing behaviour of the fluid in a flow regime at any given time. The decomposition via μ-to-σ can also relate to the behaviour where the turbulent flow disappears, if momentarily, at boundaries to let the fluid be solely governed by the laminar flow at those same boundary points. Through $\{\mu\left(t, \Psi_t, X_t; \mathbf{g}_t\right)\}_{t \in \mathbb{T}}$ and $\{\sigma\left(t, \Psi_t, X_t; \mathbf{g}_t\right)\}_{t \in \mathbb{T}}$, the interplay between different flow types and their transitions can be modified dynamically with respect to the geometry of the tunnels—e.g. by increasing or decreasing μ and σ after hitting a boundary, or entering a corridor.

3.2 Multiple Inner-Boundaries

Consider a similar particle system as before, but now, instead of having a single inner boundary, we have two inner boundaries that appear over different time horizons and different spatial positions within the master domain. This system is already more complex since the particles are being internally channeled into different spatial coordinates at different times creating an increasingly multi-layered state-space – yet, the SDE in (7) in Example 2.10 can still address this additional complexity, where we now set $m = 3$.

We fix each boundary $\{g_t^{(j)}\}_{t \in \mathbb{T}} = c^{(j)}$ for $j = 0, 1, 2, 3$ as constants, satisfying

$$-\infty < c^{(0)} < c^{(1)} < c^{(2)} < c^{(3)} < \infty.$$

For the corresponding time-segments, we additionally need to specify $\mathbb{T}^{(2)}$ for the running-time of the second internal boundary, which we choose as

$$\tau_{\text{end}}^{(1)} < \tau_{\text{start}}^{(2)} < \tau_{\text{end}}^{(2)} \le T.$$

Hence, the second internal boundary appears only after the first internal boundary ends, and divides the main tube into two tunnels back again over that non-overlapping time horizon. Now our diffusion coefficient takes the following form:

$$\sigma\left(t, \Psi_t, X_t, ; \mathbf{g}_s\right) = (c^{(0)} - X_t)(c^{(1)} - X_t + V_t)(c^{(2)} - X_t + V_t)(c^{(3)} - X_t),$$

where $\{V_t\}_{t \in \mathbb{T}}$ is continuous with $V_t = 0$ for $t \in \mathbb{T}^{(1)}$ and $t \in \mathbb{T}^{(2)}$.

Finally, we choose $\{\beta_t\}_{t \in \mathbb{T}}$ where $\beta_t(\Psi_t) = \beta_t(X(0), \ldots, X_{\tau_t})$. As such, we define it as

$$\beta_t(\Psi_t; \mathbf{g}_t) = \begin{cases} c^{(1)} & \text{for } t \in [0, \tau_{\text{start}}^{(1)}), \\[2mm] c^{(1)} + \frac{c^{(3)} - c^{(1)}}{2(\tau_{\text{end}}^{(1)} - \tau_{\text{start}}^{(1)})}(t - \tau_{\text{start}}^{(1)}) & \text{for } t \in [\tau_{\text{start}}^{(1)}, \tau_{\text{end}}^{(1)}) \text{ if } X_{\tau_{\text{start}}^{(1)}} \ge c^{(1)}, \\[2mm] c^{(1)} - \frac{c^{(1)} - c^{(0)}}{2(\tau_{\text{end}}^{(1)} - \tau_{\text{start}}^{(1)})}(t - \tau_{\text{start}}^{(1)}) & \text{for } t \in [\tau_{\text{start}}^{(1)}, \tau_{\text{end}}^{(1)}) \text{ if } X_{\tau_{\text{start}}^{(1)}} < c^{(1)}, \\[2mm] \frac{c^{(1)} + c^{(3)}}{2} & \text{for } t \in [\tau_{\text{end}}^{(1)}, \tau_{\text{start}}^{(2)}) \text{ if } X_{\tau_{\text{start}}^{(1)}} \ge c^{(1)}, \\[2mm] \frac{c^{(0)} + c^{(1)}}{2} + \frac{2c^{(2)} - (c^{(0)} + c^{(1)})}{2(\tau_{\text{start}}^{(2)} - \tau_{\text{end}}^{(1)})}(t - \tau_{\text{end}}^{(1)}) & \text{for } t \in [\tau_{\text{end}}^{(1)}, \tau_{\text{start}}^{(2)}) \text{ if } X_{\tau_{\text{start}}^{(1)}} < c^{(1)}, \\[2mm] c^{(2)} + \frac{c^{(3)} - c^{(2)}}{2(\tau_{\text{end}}^{(2)} - \tau_{\text{start}}^{(2)})}(t - \tau_{\text{start}}^{(2)}) & \text{for } t \in [\tau_{\text{start}}^{(2)}, \tau_{\text{end}}^{(2)}) \text{ if } X_{\tau_{\text{start}}^{(2)}} \ge c^{(2)}, \\[2mm] c^{(2)} - \frac{c^{(2)} - c^{(1)}}{2(\tau_{\text{end}}^{(2)} - \tau_{\text{start}}^{(2)})}(t - \tau_{\text{start}}^{(2)}) & \text{for } t \in [\tau_{\text{start}}^{(2)}, \tau_{\text{end}}^{(2)}) \text{ if } X_{\tau_{\text{start}}^{(2)}} < c^{(2)}, \\[2mm] \frac{c^{(2)} + c^{(3)}}{2} & \text{for } t \in [\tau_{\text{end}}^{(2)}, T] \quad \text{if } X_{\tau_{\text{start}}^{(2)}} \ge c^{(2)}, \\[2mm] \frac{c^{(1)} + c^{(2)}}{2} & \text{for } t \in [\tau_{\text{end}}^{(2)}, T] \quad \text{if } X_{\tau_{\text{start}}^{(2)}} < c^{(2)}, \end{cases}$$

where $\mu\left(t, \Psi_t, X_t; \mathbf{g}_t\right) = \beta_t(\Psi_t; \mathbf{g}_t) - X_t$ for $t \in \mathbb{T}$, which gives us what we need for our simulation.
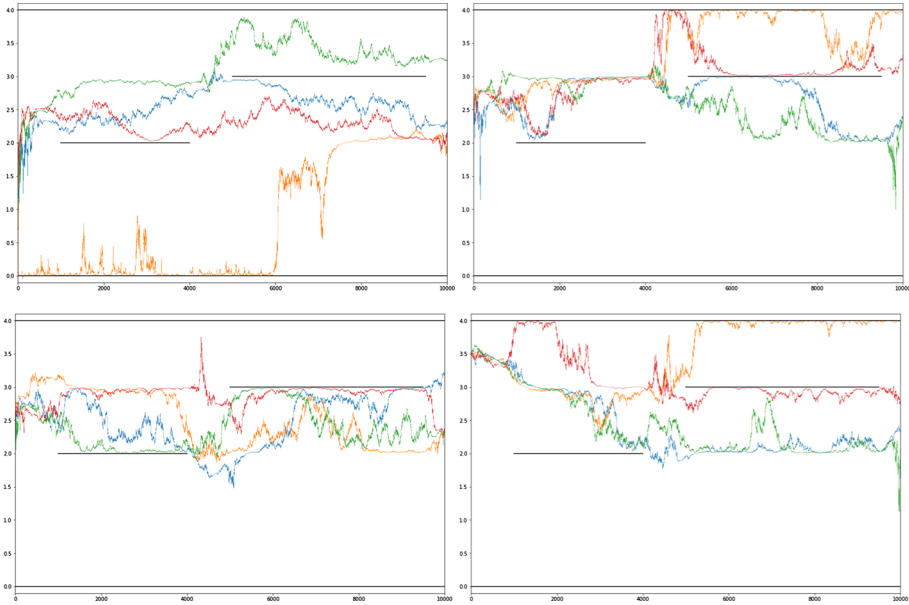
Fig. 3 $\tau_{\text{start}}^{(1)} = 1000$, $\tau_{\text{end}}^{(1)} = 4000$, $\tau_{\text{start}}^{(2)} = 5000$, $\tau_{\text{end}}^{(2)} = 9500$. Left: $x_0 = 1, 2.5$, Right: $x_0 = 2.5, 3.5$

Compared to our previous example, since we now have more internal tunnels forming the bifurcated geometry as the state-space of the particle system, the processes demonstrate more diverse tunneling effects, which can be observed in Fig. 3.

3.3 Flows in Concentric Circular Domains

We keep $m = 3$, where $\{g_t^{(j)}\}_{t \in \mathbb{T}} = c^{(j)}$ with $0 \leq c^{(0)} < c^{(1)} < c^{(2)} < c^{(3)} < \infty$. For the time-segments, we now set $\mathbb{T}^{(0)} = \mathbb{T}^{(1)} = \mathbb{T}^{(2)} = \mathbb{T}^{(4)} = \mathbb{T}$; that is, $\tau_{\text{start}}^{(j)} = 0$ and $\tau_{\text{end}}^{(j)} = T$ for $j = 0, 1, 2, 3$. Before the mathematical details, we shall first provide some samples below.

This example can be used in modelling particle systems evolving within ring-shaped geometries. Due to the way tunnels are constructed here, polar projections can never cross paths. Hence, if they represent fluid flows, they are separated in their respective corridors.

As for the mathematical model behind the samples in Fig. 4., we consider the following captives: $\{X_t^{(k,l)}\}_{t \in \mathbb{T}}$ for $k, l \in \{1, 2\}$, where

$$dX_t^{(1,1)} = \left(\frac{c^{(0)} + c^{(1)}}{2} - X_t^{(1,1)} \right) dt + (c^{(0)} - X_t^{(1,1)})(c^{(1)} - X_t^{(1,1)}) dW_t^{(1,1)}, \text{with} X_t^{(1,1)} \in [c^{(0)}, c^{(1)})$$

$$dX_t^{(1,2)} = \left(\pi - X_t^{(1,2)} \right) dt - X_t^{(1,2)}(2\pi - X_t^{(1,2)}) dW_t^{(1,2)}, \text{with} X_t^{(1,2)} \in [0, 2\pi)$$

$$dX_t^{(2,1)} = \left(\frac{c^{(2)} + c^{(3)}}{2} - X_t^{(2,1)} \right) dt + (c^{(2)} - X_t^{(2,1)})(c^{(3)} - X_t^{(2,1)}) dW_t^{(2,1)}, \text{with} X_t^{(2,1)} \in [c^{(2)}, c^{(3)})$$

$$dX_t^{(2,2)} = \left(\pi - X_t^{(2,2)} \right) dt - X_t^{(2,2)}(2\pi - X_t^{(2,2)}) dW_t^{(2,2)}, \text{with} X_t^{(2,2)} \in [0, 2\pi),$$
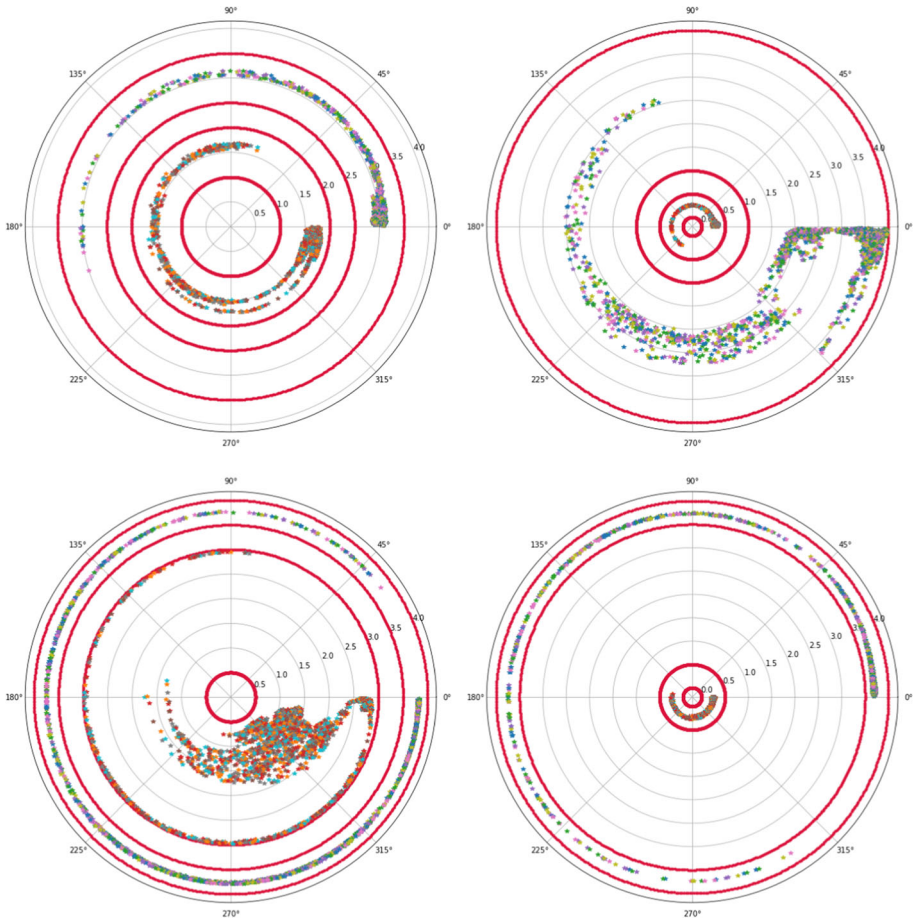
Fig. 4 Radius = 4. Top-Left: $\{c^{(0)}, c^{(1)}\} = \{1, 2\}$, $\{c^{(2)}, c^{(3)}\} = \{2.5, 3.5\}$. Top-Right: $\{c^{(0)}, c^{(1)}\} = \{0, 0.5\}$, $\{c^{(2)}, c^{(3)}\} = \{1, 4\}$. Bottom-Left: $\{c^{(0)}, c^{(1)}\} = \{0.5, 3\}$, $\{c^{(2)}, c^{(3)}\} = \{3.5, 4\}$. Bottom-Right: $\{c^{(0)}, c^{(1)}\} = \{0, 0.5\}$, $\{c^{(2)}, c^{(3)}\} = \{3.5, 4\}$

given that $\{W_t^{(k,l)}\}_{t \in \mathbb{T}}$ are $(\mathbb{P}, \{\mathcal{F}_t^X\})$-Brownian motions for $k, l \in \{1, 2\}$. Using the captives above, we construct the following paired processes $\{P_t^{(k)}\}_{t \in \mathbb{T}}$ given by:

$$P_t^{(k)} = (X_t^{(k,1)}, X_t^{(k,2)}),$$

for $k = 1, 2$ and project them on the polar coordinate system, such that $\{X_t^{(k,1)}\}_{t \in \mathbb{T}}$ models the distance from the origin and $\{X_t^{(k,2)}\}_{t \in \mathbb{T}}$ is the radian. Thus, if $r \geq c^{(3)}$ is the radius of our circular domain, we can choose $c^{(j)}$ in such ways that the paired-processes $\{P_t^{(k)}\}_{t \in \mathbb{T}}$ evolve over non-overlapping concentric circles within that domain.

4 Further Extensions

4.1 Multivariate Systems

We shall construct a multivariate setup that enables us to model interacting particle systems where each particle is a piecewise-tunneled captive process. As such, we introduce the $\mathbb{R}^n$-valued map $\mathbf{X} : \mathcal{C}(\mathbb{T} \times \mathbb{R}^n) \to \mathbb{R}^n$ written as $\{\mathbf{X}_t\}_{t \in \mathbb{T}}$, where $\mathbf{X}_t = [X_t^{(1)}, \ldots, X_t^{(i)}, \ldots, X_t^{(n)}]$ for some $n \in \mathbb{N}_+$. To specify a value for the ith coordinate, we write

$$\mathbf{X}_t[i; x] \triangleq [X_t^{(1)}, \ldots, x, \ldots, X_t^{(n)}],$$

for any $i = 1, \ldots, n$. We now work with $\mathbb{T}^{(i,j)} \subseteq \mathbb{T}$ for $j = 0 \ldots, m^{(i)}$ an $i = 1 \ldots, n$, where $m^{(i)}$ highlights that the number of time-segments across each coordinate i of $\{\mathbf{X}_t\}_{t \in \mathbb{T}}$ may be different; hence, there are $n + \sum_{i=1}^{n} m^{(i)}$ time-segments in the system. Similar to before, we define

$$\mathbb{T}^{(i,j)} = [\tau_{\text{start}}^{(i,j)}, \tau_{\text{end}}^{(i,j)}] \text{with} 0 \le \tau_{\text{start}}^{(i,j)} < \tau_{\text{end}}^{(i,j)} \le T,$$

where we set $\mathbb{T}^{(i,0)} = \mathbb{T}^{(i,m^{(i)})} = \mathbb{T}$ for all $i = 1 \ldots, n$. Accordingly, we associate a boundary function to each time-segment, which we collect in

$$\mathbf{G}_t = \{g_t^{(i,j)} : j = 0, \ldots, m^{(i)} \ \& \ i = 1, \ldots, n\}.$$

Here, it should be understood that $g^{(i,0)} \in \tilde{\mathcal{G}}^{(l)}$, $g^{(i,m^{(i)})} \in \tilde{\mathcal{G}}^{(u)}$ for all $i = 1, \ldots, n$ and $g^{(i,j)} \in \mathcal{G}$ for any i, j if $j \ne 0$ and $j \ne m^{(i)}$ when $m^{(i)} > 1$. Finally, our monitoring process $\{\boldsymbol{\Psi}_t\}_{t \in \mathbb{T}}$ is given by the following:

$$\boldsymbol{\Psi}_t = \bigcup_{i=1}^{n} \Psi_t^{(i)},$$

where each component is a random compact set defined as

$$\Psi_t^{(i)} = \left\{ X_{\tau_{\text{start}}^{(i,j)}}^{(i)} : \tau_{\text{start}}^{(i,j)} \le t, \ \text{for} \ j = 0, \ldots, m^{(i)} \right\} \text{for} i = 1, \ldots, n.$$

The notion of measurability and continuity for $\Psi_t^{(i)}$ for each $i = 1, \ldots, n$, and in turn $\boldsymbol{\Psi}_t$, should be understood as discussed in Sect. 2. In this extended setup, our multivariate piecewise-tunneled captive process $\{\mathbf{X}_t\}_{t \in \mathbb{T}}$ is governed by the following system of SDEs associated to each of its coordinates:

$$X_t^{(i)} = x_0^{(i)} + \int_0^t \mu^{(i)}(s, \boldsymbol{\Psi}_s, X_s; \mathbf{G}_s) \, ds + \int_0^t \sigma^{(i)}(s, \boldsymbol{\Psi}_s, X_s; \mathbf{G}_s) \, dM_s^{(i)},$$

with $X_0^{(i)} = x_0^{(i)} \in [g_0^{(i,0)}, g_0^{(i,m^{(i)})})$, where $\mu^{(i)}$ and $\sigma^{(i)}$ are continuous maps such that

1. $\mu^{(i)}\left(t, \boldsymbol{\Psi}_t, \mathbf{X}_t[i; g_{t-}^{(i,j)}]; \mathbf{G}_t\right) \ge dg_+^{(i,j)}(t)/dt + \Delta g_t^{(i,j)}$ if $X_{\tau_{\text{start}}^{(i,j)}}^{(i)} \ge g_{\tau_{\text{start}}^{(i,j)}}^{(i,j)}$, for any $t \in \mathbb{T}^{(i,j)}$ where $X_{t-}^{(i)} = g_{t-}^{(i,j)}$
2. $\mu^{(i)}\left(t, \boldsymbol{\Psi}_t, \mathbf{X}_t[i; g_{t-}^{(i,j)}]; \mathbf{G}_t\right) \le dg_+^{(i,j)}(t)/dt + \Delta g_t^{(i,j)}$ if $X_{\tau_{\text{start}}^{(i,j)}}^{(i)} < g_{\tau_{\text{start}}^{(i,j)}}^{(i,j)}$, for any $t \in \mathbb{T}^{(i,j)}$ where $X_{t-}^{(i)} = g_{t-}^{(i,j)}$
3. $\sigma^{(i)}\left(t, \boldsymbol{\Psi}_t, \mathbf{X}_t[i; g_{t-}^{(i,j)}]; \mathbf{G}_t\right) = 0$, for any $t \in \mathbb{T}^{(i,j)}$ where $X_{t-}^{(i)} = g_{t-}^{(i,j)}$

for every $j = 0, \ldots, m^{(i)}$ and $i = 1, \ldots, n$, where $\{M_t\}_{t \in \mathbb{T}} \in \mathcal{M}(\mathbb{T} \times \mathbb{R}^n)$. Then, we have the following result that gives the captivity property of $\{X_t\}_{t \in \mathbb{T}}$.

Proposition 4.1 *The following hold $\mathbb{P}$-almost-surely:*

1. *If $X^{(i)}_{\tau^{(i,j)}_{start}} \geq g^{(i,j)}_{\tau^{(i,j)}_{start}}$ then $X^{(i)}_t \geq g^{(i,j)}_t$ for all $t \in \mathbb{T}^{(i,j)}$*
2. *If $X^{(i)}_{\tau^{(i,j)}_{start}} < g^{(i,j)}_{\tau^{(i,j)}_{start}}$ then $X^{(i)}_t \leq g^{(i,j)}_t$ for all $t \in \mathbb{T}^{(i,j)}/\{\tau^{(i,j)}_{start}\}$*

Since Proposition 4.1 follows from similar steps as in Proposition 2.8, we omit its proof. As each coordinate is a piecewise-tunneled captive that can depend on other coordinates, $\{X_t\}_{t \in \mathbb{T}}$ allows one to integrate coupling or interaction effects between captive processes over restricted domains.

4.2 Doubly-Path-Dependent Systems

Thus far, the path-dependency of $\{X_t\}_{t \in \mathbb{T}}$ (or $\{X_t\}_{t \in \mathbb{T}}$ in the multivariate case) has been solely due to the monitoring process $\{\Psi_t\}_{t \in \mathbb{T}}$ (or $\{\Psi_t\}_{t \in \mathbb{T}}$) appearing in the coefficients of the SDEs that dynamically registers historic states of particles at corridor entry-times, which has been a key aspect to ensure internal tunneling. The construction of our stochastic processes however is flexible enough to embed a secondary layer of path-dependency into the captive dynamics through *functionals* on the paths of $\{X_t\}_{t \in \mathbb{T}}$—for the rest, we shall keep the univariate case for parsimony.

Our motivation arises from providing an alternative approach to model particle systems where their behaviour is driven both by the current state and the *memory* of the trajectory up to the present state. Note that although $\{\Psi_t\}_{t \in \mathbb{T}}$ captures glimpses of memory at discrete time points, it still does not account for the full historic path information. Systems with memory appear in numerous areas such as open-system oscillators [60], Dirac particles in electromagnetic fields [61], condensed matter dynamics in chemical physics [62], thermodynamics of materials [63, 64], media propogation with viscoelasticity [65], particles in turbulent flow [59, 66] and econophysics [67, 68] to name a few. Below, we shall demonstrate how our framework can address enriched memory effects in confined dynamics by leveraging the deep structure of *functional* SDEs that communicate naturally with piecewise-tunneled captive processes.

First, we shall on-board additional mathematical machinery. Until now, we have used the notation X_t or $X(t)$ to denote the value of X at time $t \in \mathbb{T}$ and $\{X_t\}_{t \in \mathbb{T}}$ as the process. We shall now introduce an additional notation to denote *subsections* of the paths of $\{X_t\}_{t \in \mathbb{T}}$ as follows:

$$X_{[0,t]} = \{X_s : 0 \leq s \leq t\}.$$

Hence, $X_{[0,t]}$ is a collection of all the values of the process $\{X_t\}_{t \in \mathbb{T}}$ from time 0 until time $t \leq T$, representing a part of a path of the process. Accordingly, $X_\mathbb{T}$ denotes the *full* path of the captive process over its entire lifetime. We shall now provide the following definition (see, for example, [69]).

Definition 4.2 A non-anticipative functional on $\mathcal{C}(\mathbb{T} \times \mathbb{R})$ is a measurable map $F : \Lambda \to \mathbb{R}$ on the vector bundle

$$\Lambda_\mathbb{T} \triangleq \bigcup_{t \in \mathbb{T}} \mathcal{C}([0, t] \times \mathbb{R}).$$

As such, denote $\Lambda_{[0,t]} = \mathcal{C}([0, t] \times \mathbb{R})$ so that $\Lambda_\mathbb{T} = \bigcup_{t \in \mathbb{T}} \Lambda_{[0,t]}$.

Note that the SDE (5) in Definition 2.6 asks the piecewise-tunneled captive process $\{X_t\}_{t \in \mathbb{T}}$ to be continuous by imposing continuity conditions on the coefficients μ and σ and by choosing $\{M_t\}_{t \in \mathbb{T}} \in \mathcal{M}(\mathbb{T} \times \mathbb{R})$. Since the continuous-path property of $\{X_t\}_{t \in \mathbb{T}}$ plays a key role in our proofs of captivity (although we can envision discontinuous extensions), we shall make sense of what it means for the *functional* F in Definition 4.2 to be continuous. As a start, each $\Lambda_{0,t}$ can be equipped with the supremum norm $||.||_\infty$. Since the paths in $\Lambda_\mathbb{T}$ can be of various lengths, a stopped-path approach is adopted in [70] to define the distance of such paths under $||.||_\infty$, based on a horizontal extension map

$$X_{[0,t]} \in \Lambda_{[0,t]} \mapsto X_{[0,t+\{h\}]} \in \Lambda_{[0,t+h]}, \tag{10}$$

where we have

$$X_{[0,t+\{h\}]}(u) = \begin{cases} X(u), & \text{for } u \in [0, t] \\ X(t), & \text{for } u \in (t, t + h], \end{cases}$$

for $h \geq 0$. Hence, the path is *stretched* in the direction of time using the last value of the path – this is the reason we use a special notation $X_{[0,t+\{h\}]}$ instead of $X_{[0,t+h]}$ in the context of horizontal extension, since $X_{[0,t+h]}$ would be the actual realization of the path over $[0, t + h]$ instead of the stretched version of the path. Now, for another path $X'_{[0,t]}$, a distance on $\Lambda_\mathbb{T}$ can be defined using the horizontal extension apparatus given by the following norm:

$$D_\infty(X_{[0,t]}, X'_{[0,s]}) = ||X_{[0,t+\{s-t\}]} - X'_{[0,s]}||_\infty + s - t$$

for $t \leq s$, which we shall refer to as the Dupire metric. Thus, we can work on the metric space $(\Lambda_\mathbb{T}, D_\infty)$, and the continuity of a functional with respect to $D_\infty(.)$ can be defined. More specifically, from [70], we adopt the following definition.

Definition 4.3 A non-anticipative functional $F : \Lambda_\mathbb{T} \to \mathbb{R}$ is Λ-continuous at $X_{[0,t]} \in \Lambda_{[0,t]}$, if $\forall \epsilon > 0, \exists \delta > 0$ such that for all $X'_{[0,s]} \in \Lambda_\mathbb{T}$, we have

$$D_\infty(X_{[0,t]}, X'_{[0,s]}) < \delta \Rightarrow |F(X_{[0,t]}) - F(X'_{[0,s]})| < \epsilon.$$

The functional $F : \Lambda_\mathbb{T} \to \mathbb{R}$ is fully Λ-continuous if it is Λ-continuous at all $X_{[0,t]} \in \Lambda_{0,t}$.

For what follows, the space of non-anticipative functionals on $\Lambda_\mathbb{T}$ that are fully Λ-continuous as in Definition 4.3 and bounded (where we refer to [69] for the definition of boundedness-preserving functionals) is denoted as $\mathbb{C}_b(\Lambda_\mathbb{T})$.

Example 4.4 If $F(X_{[0,t]}) = \max(\{X_s : 0 \leq s \leq t\})$, then $F \in \mathbb{C}_b(\Lambda_\mathbb{T})$. Similarly, if $F(X_{[0,t]}) = \min(\{X_s : 0 \leq s \leq t\})$, then $F \in \mathbb{C}_b(\Lambda_\mathbb{T})$.

Remark 4.5 Using the pathwise-regularity result in [69], if $F \in \mathbb{C}_b(\Lambda_\mathbb{T})$, then $\{Z_t\}_{t \in \mathbb{T}}$ defined by $Z_t = F(X_{[0,t]})$ is a continuous process, given that $\{X_t\}_{t \in \mathbb{T}}$ is continuous.

We are now in the position to provide the following definition.

Definition 4.6 Keep the setup as above. Then, a *doubly-path-dependent piecewise-tunneled captive process* $\{X_t\}_{t \in \mathbb{T}} \in \mathcal{C}(\mathbb{T} \times \mathbb{R})$ is the solution to an SDE governed by

$$X_t = x_0 + \int_0^t \mu\left(s, \Psi_s, F\left(X_{[0,s]}\right), X_s; \mathbf{g}_s\right) ds + \int_0^t \sigma\left(s, \Psi_s, H\left(X_{[0,s]}\right), X_s; \mathbf{g}_s\right) dM_s, \tag{11}$$

with $X_0 = x_0 \in [g_0^{(0)}, g_0^{(m)})$, where $F, H \in \mathbb{C}_b(\Lambda_\mathbb{T})$, and μ and σ are continuous maps satisfying

1. $\mu\left(t, \Psi_t, F\left(X_{[0,t]}\right), g_{t-}^{(j)}; \mathbf{g}_t\right) \geq \mathrm{d}g_+^{(j)}(t)/\mathrm{d}t + \Delta g_t^{(j)}$ if $X_{\tau_{\mathrm{start}}^{(j)}} \geq g_{\tau_{\mathrm{start}}^{(j)}}^{(j)}$, for any $t \in \mathbb{T}^{(j)}$

   where $X_{t-} = g_{t-}^{(j)}$

2. $\mu\left(t, \Psi_t, F\left(X_{[0,t]}\right), g_{t-}^{(j)}; \mathbf{g}_t\right) \leq \mathrm{d}g_+^{(j)}(t)/\mathrm{d}t + \Delta g_t^{(j)}$ if $X_{\tau_{\mathrm{start}}^{(j)}} < g_{\tau_{\mathrm{start}}^{(j)}}^{(j)}$, for any $t \in \mathbb{T}^{(j)}$

   where $X_{t-} = g_{t-}^{(j)}$

3. $\sigma\left(t, \Psi_t, F\left(X_{[0,t]}\right), g_{t-}^{(j)}; \mathbf{g}_t\right) = 0$, for any $t \in \mathbb{T}^{(j)}$ where $X_{t-} = g_{t-}^{(j)}$

for every $j = 0, \ldots, m$, where $\{M_t\}_{t \in \mathbb{T}} \in \mathcal{M}(\mathbb{T} \times \mathbb{R})$.

From Definition 4.6, we can prove the same captivity property as given in Proposition 2.8 (or Proposition 4.1 in the multivariate case). Such an extension opens up new avenues for applications, where captive processes may have dynamic interactions with their historic captive paths.

4.3 Doubly-Stochastic Systems

Thus far, we kept boundaries fairly arbitrary as long as they belong to $\tilde{\mathcal{G}}^{(l)}$, $\tilde{\mathcal{G}}^{(u)}$ or $\mathcal{G}$. We shall briefly discuss a doubly-stochastic system, whereby the internal tunneling boundaries are randomly generated by a pure-jump stochastic process with càdlàg paths and finite activity.

Definition 4.7 Let $\{P_t\}_{t \in \mathbb{T}}$ be an adapted pure-jump stochastic process, which produces the internal tunnels whenever $P_t \in (g_t^{(l)}, g_t^{(u)})$, given that $g^{(l)} \in \tilde{\mathcal{G}}^{(l)}$ and $g^{(u)} \in \tilde{\mathcal{G}}^{(u)}$ are the lower and upper master boundaries, respectively.

Note that we cannot index master boundaries numerically as before, since we do not know what m would be *a priori*. Moreover, since internal tunnels are now randomly generated, we no longer have *pre-defined* $\mathbb{T}^{(j)}$s associated to the internal tunnels as neither $\tau_{\mathrm{start}}^{(j)}$ nor $\tau_{\mathrm{end}}^{(j)}$ are known in advance. If also the jump magnitudes of $\{P_t\}_{t \in \mathbb{T}}$ are random, then we do not even know where the tunnels would be *located* a priori. These additional complexities from a doubly-stochastic framework make explicit constructions challenging. Nonetheless, we shall produce an example of such a system below.

4.3.1 Random Tunnel Formation

Let $\{P_t\}_{t \in \mathbb{T}}$ be a counting process such that $\Delta P_t \in \{0, c\}$ for some positive integer $c > 0$, where $P_0 = g_0^{(l)} = g_t^{(l)} = 0$ for any $t \in \mathbb{T}$. We also choose $g^{(u)}$ constant such that $\{g_t^{(u)}\}_{t \in \mathbb{T}} = k * c$ for some other integer $k \geq 2$ so that if $\{P_t\}_{t \in \mathbb{T}}$ jumps before T, there is at least one internal tunnel boundary formed. We denote $\tau^{(j)}$ as the jth jump time of $\{P_t\}_{t \in \mathbb{T}}$ so that

$$\mathbb{T}^{(j)} = [\tau^{(j)}, \tau^{(j+1)}) \mathrm{with} \tau^{(j)} < \tau^{(j+1)}, \tag{12}$$

for $j = 1, \ldots$ – even if we do not know the jump times of $\{P_t\}_{t \in \mathbb{T}}$ in advance, we know that *if* there is a jump, it will be of size c, which helps to define the μ and σ of our $\{X_t\}_{t \in \mathbb{T}}$ in (5). On top of this, we need the jump times to be adapted to the filtration of $\{X_t\}_{t \in \mathbb{T}}$, so that when a jump of $\{P_t\}_{t \in \mathbb{T}}$ occurs, $\{X_t\}_{t \in \mathbb{T}}$ is informed of this occurrence at that very time – hence, we extend the SDE coefficients in (5) to include $\{P_t\}_{t \in \mathbb{T}}$ such that $\tau^{(j)}$s become $\{\mathcal{F}_t^X\}_{t \in \mathbb{T}}$-stopping times:

$$X_t = x_0 + \int_0^t \mu\left(s, \Psi_s, P_s, X_s; \mathbf{g}_s\right) \mathrm{d}s + \int_0^t \sigma\left(s, \Psi_s, P_s, X_s; \mathbf{g}_s\right) \mathrm{d}M_s.$$

More specifically, we can choose some continuous function σ with zeros at $\{0, c, \ldots, k\}$, and let $\mu\left(t, \Psi_t, P_t, X_t; \mathbf{g}_t\right) = \beta_t(\Psi_t, P_t; \mathbf{g}_t) - X_t$ as before, where for some small $\epsilon > 0$, we have the following function that extends as more stopping-times occur in the logic outlined below:

$$
\beta_t(\Psi_t, P_t; \mathbf{g}_t) = \begin{cases}
c & \text{for } t \in [0, T \wedge \tau^{(1)}], \\[2mm]
c - \frac{c(t-\tau^{(1)})}{2(T-\tau^{(1)})} & \text{for } t \in [\tau^{(1)}, T] \text{ if } X_{\tau^{(1)}} < c \text{ and } \tau^{(1)} < T, \\[2mm]
c + \frac{c(t-\tau^{(1)})}{2\epsilon} & \text{for } t \in [\tau^{(1)}, T \wedge \tau^{(1)} + \epsilon] \text{ if } X_{\tau^{(1)}} \geq c \text{ and } \tau^{(1)} < T, \\[2mm]
\frac{3}{2}c & \text{for } t \in [\tau^{(1)} + \epsilon, T \wedge \tau^{(2)}] \text{ if } X_{\tau^{(1)}} \geq c \text{ and } \tau^{(2)} \leq T, \\[2mm]
\frac{3}{2}c - \frac{c(t-\tau^{(2)})}{2(T-\tau^{(2)})} & \text{for } t \in [\tau^{(2)}, T] \text{ if } X_{\tau^{(1)}} \geq c, \ X_{\tau^{(2)}} < \frac{3}{2}c \text{ and } \tau^{(2)} < T, \\[2mm]
\frac{3}{2}c + \frac{c(t-\tau^{(2)})}{2\epsilon} & \text{for } t \in [\tau^{(2)}, T \wedge \tau^{(2)} + \epsilon] \text{ if } X_{\tau^{(1)}} \geq c, \ X_{\tau^{(2)}} \geq \frac{3}{2}c \text{ and } \tau^{(2)} < T, \\[2mm]
\vdots &
\end{cases}
$$

This function expands in the pattern above depending on $\{P_t\}_{t \in \mathbb{T}}$ and what $k * c$ is. We have to highlight that this function falls apart if there is a j where the following happens:

$$
\tau^{(j+1)} - \tau^{(j)} < \epsilon.
$$

Therefore, unless $\{P_t\}_{t \in \mathbb{T}}$ is a process where $\mathbb{P}(\tau^{(j+1)} - \tau^{(j)} < \epsilon) = 0$ for every j, this example would work only for a subsample of all possible trajectories that $\{P_t\}_{t \in \mathbb{T}}$ may take where $(\tau^{(j+1)} - \tau^{(j)}) > \epsilon$ happens to hold. Hence, even for such a simple example, the doubly-stochastic nature may block $\{X_t\}_{t \in \mathbb{T}}$ to be a piecewise-tunneled captive process. Therefore, in the example above, $\{P_t\}_{t \in \mathbb{T}}$ needs to satisfy the following law:

$$
\mathbb{P}(P_{t+\epsilon} - P_t > 0) = 0,
$$

for all $t \in \mathbb{T}$ for a given $\epsilon > 0$, so that $\{X_t\}_{t \in \mathbb{T}}$ is a piecewise-tunneled captive process $\mathbb{P}$-a.s. We shall leave a more detailed account of such doubly-stochastic formations for future.

5 Conclusion

We introduced a new family of stochastic processes we call *piecewise-tunneled captive processes* that extend the captive diffusions of [19] through path-dependent dynamics, which in turn governs the process also to remain within internal tunnels. We believe that our framework can be applied in modelling random flows in fluid dynamics and random particle systems exposed to channeling effects. We constructed explicit examples and provided simulations for demonstration. We also provided extensions to the framework to expand on new modelling capabilities for applications.

This work can further be extended in several ways. Amongst several possible theoretical investigations, distributions of the first hitting-times to boundaries (master and tunnels) of various subclasses of piecewise-tunneled captive processes can be studied. On the other hand, calibration studies based on empirical data would be useful and insightful. Accordingly, one

can continue with approximating the distribution of finding a particle in any possible corridor at any given time by sampling a large number of paths under a calibrated model. In practice, one can also formulate stochastic control problems whereby one aims to design optimal tunnels to maximize corresponding objectives (e.g. in fluid dynamics). In addition, these processes can also be utilized in constructing Hermitian-valued evolutions that are captive under the Loewner-order in the spirit of [23] to model stochastic covariance structures that may be 'tunneled' with respect to different regimes.

Data Availability Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

References

1. Itô, K.: On stochastic differential equations on a differentiable manifold I. N. Math. J. **1**, 35–47 (1950)
2. Skorokhod, A.V.: Stochastic equations for diffusion processes in a bounded region I. Theory Probab. Appl. **6**, 287–298 (1961)
3. Durrett, R.T., Iglehart, D.L.: Functionals of Brownian Meander and Brownian Excursion. Ann. Probab. **5**, 130–135 (1977)
4. Pitman, J., Yor, M.: Bessel processes and infinitely divisible laws. In: Williams, D. (ed.) Stochastic Integrals, vol. 851. Springer, Berlin (1980)
5. Harrison, J.M., Reiman, M.I.: On the distribution of multidimensional reflected Brownian Motion. SIAM J. Appl. Math. **41**, 345–361 (1981)
6. Ricciardi, L.M., Sacordote, L.: On the probability densities of an Ornstein-Uhlenbeck process with a reflecting boundary. J. Appl. Probab. **24**(2), 355–369 (1987)
7. Emery, M.: Stochastic Calculus in Manifolds. Springer, Berlin (1989)
8. Asmussen, S., Glynn, P., Pitman, J.: Discretization error in simulation of one- dimensional reflecting Brownian Motion. Ann. Appl. Probab. **5**, 875–896 (1995)
9. Inoue, J., Sato, S., Ricciardi, L.M.: A note on the moments of the first-passage time of the Ornstein-Uhlenbeck process with a reflecting boundary. Richerce Mat. **46**, 87–99 (1997)
10. Ward, A.R., Glynn, P.W.: Properties of the reflected Ornstein-Uhlenbeck process. Queuing Syst. **44**(2), 109–123 (2003)
11. Obloj, J., Yor, M.: An explicit Skorokhod embedding for the age of Brownian excursions and Azéma Martingale. Stoch. Process. Appl. **110**, 83–110 (2004)
12. Katori, M., Tanemura, H.: Symmetry of matrix-valued stochastic processes and noncolliding diffusion particle systems. J. Math. Phys. **45**, 3058–3085 (2004)
13. Linetsky, V.: On the transition densities for reflected diffusions. Adv. Appl. Probab. **37**(2), 435–460 (2005)
14. Deuschel, J.-D., Zambotti, L.: Bismut-Elworthy's formula and random walk representation for SDEs with reflection. Stoch. Process. Appl. **115**, 907–925 (2005)
15. Bo, L., Zhang, L., Wang, Y.: On the first-passage times of reflected OU processes with two-sided barriers. Queuing Syst. **54**(4), 313–316 (2006)
16. Yen, J.Y., Yor, M.: Local Times and Excursion Theory for Brownian Motion. Springer, Berlin (2013)
17. Katori, M.: Determinantal Martingales and noncolliding diffusion processes. Stoch. Process. Appl. **124**, 3724–3768 (2014)

18. Pitman, J., Winkel, M.: Squared bessel processes of positive and negative dimension embedded in Brownian local times. Electron. Commun. Probab. **23**, 1–13 (2018)

19. Mengütürk, L.A., Mengütürk, M.C.: Captive diffusions and their applications to order-preserving dynamics. Proc. R. Soc. A: Math. Phys. Eng. Sci. **476**, 1–18 (2020)

20. Dyson, F.J.: A Brownian-Motion model for the eigenvalues of a random matrix. J. Math. Phys. **3**, 1191–1198 (1962)

21. Pal, S., Pitman, J.: One-dimensional Brownian particle systems with rank-dependent drifts. Ann. Appl. Probab. **18**, 2179–2207 (2008)

22. Shkolnikov, M.: Large systems of diffusions interacting through their ranks. Stoch. Process. Appl. **122**, 1730–1747 (2012)

23. Mengütürk, L.A., Mengütürk, M.C.: *From Loewner-Captive Hermitian Diffusions to Risk-Captive Efficient Frontiers*, Working paper (2022)

24. Gemmell, D.S.: Channeling and related effects in the motion of charged particles through crystals. Rev. Mod. Phys. **46**, 129–227 (1974)

25. Belluci, S., Biryukov, V.M., Cordelli, A.: Channeling of high-energy particles in a multi-wall nanotube. Phys. Lett. B **608**, 53–58 (2005)

26. Kozlov, A., Shulga, N., Cherkaskiy, V.: Spectral method in quantum theory of channeling phenomena of fast charged particles in crystals. Phys. Lett. A **374**, 4690–4694 (2010)

27. Petrovic, S., Neskovic, N., Berec, V., Cosic, M.: Superfocusing of channeled protons and subatomic measurement resolution. Phys. Rev. A **85**, 032901 (2012)

28. Shulga, N.F., Shulga, S.N.: Geometrical optics method in the theory of channeling of high energy particles in crystals. Phys. Lett. B **791**, 225–229 (2019)

29. Cipriano, F., Cruzeiro, A.B.: Navier-Stokes equation and diffusions on the group of homeomorphisms of the torus. Commun. Math. Phys. **275**, 255–269 (2007)

30. Constantin, P., Iyer, G.: A stochastic Lagrangian representation of the three- dimensional incompressible Navier-Stokes equations. Commun. Pure Appl. Math. **61**, 330–345 (2008)

31. Gliklikh, Y.E.: Solutions of Burgers, Reynolds, and Navier-stokes equations via stochastic perturbations of inviscid flows. J. Nonlinear Math. Phys. Suppl. **17**, 15–29 (2010)

32. Arnaudon, M., Chen, X., Cruzeiro, A.B.: Stochastic Euler-Poincaré reduction. J. Math. Phys. **55**, 081507 (2014)

33. Holm, D.D.: Variational principles for stochastic fluid dynamics. Proc. R. Soc. A: Math. Phys. Eng. Sci. **471**, 1–19 (2015)

34. Carlo, D.D., Irimia, D., Tompkins, R.G., Toner, M.: Continuous intertial focusing, ordering, and separation of particles in microchannels. Proc. Natl. Acad. Sci. USA **104**, 18892–18897 (2007)

35. Baek, S.G., Park, S.: Effect of wall distance on the prediction of variable property flow with two-equation turbulence models. Int. J. Comput. Fluid Dyn. **19**, 447–455 (2005)

36. Kolomeisky, A.B., Uppulury, K.: How interactions control molecular transport in channels. J. Stat. Phys. **142**, 1268–1276 (2011)

37. Kim, C., Karniadakis, G.E.: Brownian motion of a Rayleigh particle confined in a channel: a generalised Langevin equation approach. J. Stat. Phys. **158**, 1100–1125 (2015)

38. Himmelberg, C.J.: Measurable relations. Fundam. Math. **87**, 53–72 (1975)

39. Castaing, C., Valadier, M.: Convex Analysis and Measurable Multifunctions. Lecture Notes in Mathematics, vol. 580. Springer, Berlin (1977)

40. Molchanov, I.: Theory of Random Sets. Springer, Berlin (2005)

41. Nguyen, H.T.: An Introduction to Random Sets. Chapman and Hall/CRC Press, London (2006)

42. Schmelzer, B.: On solutions of stochastic differential equations with parameters modeled by random sets. Int. J. Approx. Reason. **51**, 1159–1171 (2010)

43. Rogers, L.C.G., Williams, D.: Diffusions, Markov Processes and Martingales, vol. II. Cambridge Mathematical Library, Cambridge (2000)

44. Øksendal, B.K.: Stochastic Differential Equations: An Introduction with Applications. Springer, Berlin (2003)

45. Giorno, V., Nobile, A.G., Ricciardi, L.M.: On some diffusion approximations to queuing systems. Adv. Appl. Probab. **18**, 991–1014 (1986)

46. Mishura, Y., Yurchenko-Tytarenko, A.: Standard and fractional reflected Ornstein- Uhlenbeck processes as the limits of square roots of cox-ingersoll-ross processes. Stoch. Int. J. Probab. Stoch. Process. (2022). https://doi.org/10.1080/17442508.2022.2047188

47. Goldstein R.S., Keirstead, W.P.: On the term structure of interest rates in the presence of reflecting and absorbing boundaries. SSRN Electron. J. (1997)

48. Kuan, G.C.H., Webber, N.: Pricing barrier options with one-factor interest rate models. J. Deriv. **10**, 33–50 (2003)

49. Bo, L., Wang, Y., Yang, X.: Some integral functionals of reflected SDEs and their applications in finance. Quant. Financ. **11**(3), 343–348 (2008)
50. Ricciardi, L.M.: Stochastic Population Theory: Diffusion Processes in Mathematical Ecology. Springer, Berlin (1986)
51. Aalen, O.O., Gjessing, H.K.: Survival models based on the Ornstein-Uhlenbeck processes. Lifetime Data Anal. **10**, 407–423 (2004)
52. Zang, Q., Zhang, L.: A general lower bound of parameter estimation for reflected Ornstein-Uhlenbeck processes. J. Appl. Probab. **53**(1), 22–32 (2016)
53. Lee, C., Song, J.: On drift parameter estimation for reflected fractional Ornstein-Uhlenbeck processes. Stoch. Int. J. Probab. Stoch. Process. **88**, 751–778 (2016)
54. Danelon, C., Brando, T., Winterhalter, M.: Probing the orientation of reconstituted Maltoporin channels at the single-protein level. J. Biol. Chem. **278**, 35542–35551 (2003)
55. Schwarz, G., Danelon, C., Winterhalter, M.: On translocation through a membrane channel via an internal binding site: kinetics and voltage dependence. Biophys. J . **84**, 2990–2998 (2004)
56. Graaf, D.B., Eaton, J.K.: Reynolds-number scaling of the flat-plate turbulent boundary layer. J. Fluid Mech. **422**, 319–346 (2000)
57. Govindarajan, R., L'vov, V.S., Procaccia, I.: Stabilization of hydrodynamic flows by small viscosity variations. Phys. Rev. E 67, 026310 (2003)
58. Pivkin, I.V., Karniadakis, G.: Controlling density fluctuations in wall-bounded dissipative particle dynamics systems. Phys. Rev. Lett. **96**, 206001 (2006)
59. Schmidt, J.R., Wendt, J.O.L., Kerstein, A.R.: Non-equilibrium wall deposition of inertial particles in turbulent flow. J. Stat. Phys. **137**(2), 233 (2009)
60. Tarasov, V.E.: The fractional oscillator as an open system. Cent. Eur. J. Phys. **10**(2), 382–389 (2012)
61. Tarasov, V.E.: Dirac particle with memory: proper time non-locality. Phys. Lett. A **384**, 126303 (2020)
62. Evans, M.W., Grigolini, P., Parravicini, G.P.: Memory Function Approaches to Stochastic Problems in Condensed Matter. Intersience, De Gruyter, NY (1985)
63. Day, W.A.: The Thermodynamics of Simple Materials with Fading Memory. Springer, Berlin (1972)
64. Amendola, G., Fabrizio, M., Golden, J.M.: Thermodynamics of Materials with Memory: Theory and Applications. Springer, Berlin (2011)
65. Mainardi, F.: Fractional Calculus and Waves Linear Viscoelasticity: An Introduction to Mathematical Models. Imperial College Press, London (2010)
66. Shin, M., Lee, J.W.: Memory effect in the Eulerian particle deposition in a fully developed turbulent channel flow. J. Aerosol Sci. **32**, 675–693 (2001)
67. Tarasov, V.E., Tarasova, V.V.: Time-dependent fractional dynamics with memory in quantum and economic physics. Ann. Phys. **383**, 579–599 (2017)
68. Tarasova, V.V., Tarasov, V.E.: Concept of dynamic memory in economics. Commun. Nonlinear Sci. Numer. Simul. **55**, 127–145 (2018)
69. Cont, R., Fournie, D.-A.: Change of variable formulas for non-anticipative functionals on path space. J. Funct. Anal. **259**, 1043–1072 (2010)
70. Dupire, B.: Functional Ito Calculus. Portfolio Research Paper, Bloomberg (2009)