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Research Article

On the density of branching Brownian motion

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Abstract

We consider a d-dimensional dyadic branching Brownian motion, and study the density of its support in the region where there is typically exponential growth of particles. Using geometric arguments and an extension of a previous result on the probability of absence of branching Brownian motion in linearly moving balls of fixed size, we obtain sharp asymptotic results on the covering radius of the support of branching Brownian motion, which is a measure of its density. As a corollary, we obtain large deviation estimates on the volume of the $r(t)$-enlargement of the support of branching Brownian motion when $r(t)$ decays exponentially in time t. As a by-product, we obtain the lower tail asymptotics for the mass of branching Brownian motion falling in linearly moving balls of exponentially shrinking radius, which is of independent interest.

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1. Introduction

The model studied in this paper is a branching Brownian motion (BBM) evolving in $\mathbb{R}^d$. It is well-known that typically the *mass*, i.e., the number of particles, of a BBM grows exponentially in time. To be precise, if N_t denotes the total mass of a strictly dyadic BBM at time t and β is the branching rate, then $N = (N_t)_{t\geq 0}$ is a Yule process, and the limit

$$M := \lim_{t\to\infty} N_t\, e^{-\beta t}$$

exists and is positive almost surely. It is also known [3, 4, 13] that the *speed* of a strictly dyadic BBM is $\sqrt{2\beta}$, which means that typically for large time the support of BBM at time t is contained in $B(0, \sqrt{2\beta}(1+\varepsilon)t)$, where we use $B(x,r)$ to denote the open ball of radius r and center x, but not contained in $B(0, \sqrt{2\beta}(1-\varepsilon)t)$ for any $0 < \varepsilon < 1$. Moreover, $\mathcal{B}_t := B(0, \sqrt{2\beta}(1-\varepsilon)t)$ is a region where there is typically exponential growth of particles. Then, a natural question concerns the spatial distribution of mass at time t: how homogeneously are the exponentially many particles spread out over $\mathcal{B}_t$? If they are spread out sufficiently homogeneously, then one may formulate this in terms of the density of the support of BBM, and obtain quantitative results on its covering radius in $\mathcal{B}_t$. This work presents fine results on the geometry of particles in a BBM at time t for large t, and

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mainly aims at answering the question of how dense the support of BBM is in the region where there is typically exponential growth of particles.

We first extend a previous result [14, Thm. 1] on the probability of aytpically small mass of BBM inside moving balls of fixed radius to moving balls of time-dependent radius; then using this extension and some geometric arguments, we obtain a large deviation (LD) result on the asymptotic behavior of the density of the support of BBM. As a corollary, we show that for a suitably decreasing function $r : \mathbb{R}_+ \to \mathbb{R}_+$, the $r(t)$-enlargement of the support of BBM at time t fills up $\mathcal{B}_t$ with overwhelming probability as $t \to \infty$.

1.1. Formulation of the problem

Let $Z = (Z(t))_{t\geq 0}$ be a d-dimensional strictly dyadic BBM with constant branching rate $\beta > 0$. Here, t represents time, and strictly dyadic means that every time a particle branches, it gives exactly two offspring. The process starts with a single particle, which performs a Brownian motion in $\mathbb{R}^d$ for a random exponential time of parameter β, at which the particle dies and simultaneously gives birth to two offspring. Similarly, starting from the position where their parent dies, each offspring particle repeats the same procedure as their parent independently of others and of the parent, and the process evolves in time in this way. The Brownian motions and exponential lifetimes of particles are all independent from one another. For each $t \geq 0$, $Z(t)$ can be viewed as a discrete measure on $\mathbb{R}^d$. Let P_x and E_x, respectively, denote the probability and corresponding expectation for Z when the process starts with a single particle at position $x \in \mathbb{R}^d$, that is, when $Z(0) = \delta_x$, denoting the Dirac measure at x. When $Z(0) = \delta_0$, we simply use P and E. For a Borel set $B \subseteq \mathbb{R}^d$ and $t \geq 0$, we write $Z_t(B)$ to denote the mass of Z that fall inside B at time t. We write $N_t := Z_t(\mathbb{R}^d)$ for the total mass at time t. The *range* of Z up to time t, and the full range of Z, are defined respectively as

$$R(t) = \bigcup_{0\leq s\leq t} \mathrm{supp}(Z(s)), \qquad R = \bigcup_{t\geq 0} R(t). \tag{1.1}$$

By the classical result of [13], it is well-known that the *speed* of strictly dyadic BBM in one dimension is equal to $\sqrt{2\beta}$, which was later generalized to higher dimensions by [8]. More precisely, we have the following result.

Theorem A (Speed of BBM; [8, 13]). Let Z be a strictly dyadic BBM with branching rate $\beta > 0$ in $\mathbb{R}^d$. For $t \geq 0$ define $M_t := \inf\{r > 0 : \mathit{supp}(Z(t)) \subseteq B(0, r)\}$ to be the radius of the minimal ball that contains the support of BBM at time t. Then, in any dimension,

$$M_t/t \to \sqrt{2\beta} \quad \text{in probability} \quad \text{as} \quad t \to \infty.$$

Note that M_t quantifies the spatial spread of BBM at time t so that M_t/t is a measure of the speed of BBM. More sophisticated results on the speed of BBM, such as almost sure results and higher order sublinear corrections, exist in the literature (see for example [3, 11, 12]). For our purposes, Theorem A suffices; it says that typically for large t and any $\varepsilon > 0$, at time t there will be particles outside $B(0, \sqrt{2\beta}(1-\varepsilon)t)$ but no particles outside $B(0, (\sqrt{2\beta}(1+\varepsilon)t)$. Therefore, when we study the density of the support of BBM at time t, to obtain meaningful results, we consider the density within a *subcritical ball*, which we define as follows.

Definition 1.1 (Subcritical ball). We call $B = (B(0, \rho_t))_{t\geq 0}$ a *subcritical ball* if there exists $0 < \varepsilon < 1$ and time t_0 such that $B(0, \rho_t) \subseteq B\left(0, \sqrt{2\beta}(1-\varepsilon)t\right)$ for all $t \geq t_0$.

Remark 1.2. We emphasize that the term 'subcritical' above is a property of the radius ρ_t, which can be rewritten as $\limsup_{t\to\infty} \frac{\rho_t}{t} < \sqrt{2\beta}$.

Also, we use the term *subcritical ball* both in the sense of a time-dependent ball $B = (B(0, \rho_t))_{t\geq 0}$ as in Definition 1.1, and also simply as a snapshot taken of a time-dependent ball at a fixed large time t as $B(0, \rho_t)$.

Typically, for a unit vector $\mathbf{e}$, fixed radius $r_0 > 0$, and $0 \leq \theta < 1$, the mass of BBM falling in $B_t := B(\theta\sqrt{2\beta}t\mathbf{e}, r_0)$ at time t is $\exp[\beta(1-\theta^2)t + o(t)]$ as $t \to \infty$. This follows from [2, Corollary 4], which implies that

$$\lim_{t\to\infty} \frac{1}{t} \log Z_t(B_t) = \beta(1-\theta^2) \quad \text{a.s.}$$

The current work is motivated by, and presents applications of the following LD result from [14], which concerns atypically small mass falling in linearly moving balls of fixed radius.

Theorem B (Lower tail asymptotics for mass inside a moving ball; [14])**.** Let $0 \leq \theta < 1$, $r_0 > 0$, and $\mathbf{e}$ be a unit vector in $\mathbb{R}^d$. For $t \geq 0$, define $B_t := B(\theta\sqrt{2\beta}t\mathbf{e}, r_0)$. Then, for $0 \leq a < 1-\theta^2$, in any dimension $d \geq 1$,

$$\lim_{t\to\infty} \frac{1}{t} \log P\left(Z_t(B_t) < e^{\beta a t}\right) = -\beta \times J, \tag{1.2}$$

where $J = J(\theta, a)$ is a positive rate function.* When $a = 0$, as a special case,

$$\lim_{t\to\infty} \frac{1}{t} \log P\left(Z_t(B_t) = 0\right) = -\beta \times J(\theta, 0) = -2\beta(\sqrt{2}-1)(1-\theta). \tag{1.3}$$

Observe that (1.3) gives the large-time behavior of the probability of absence of Z in linearly moving balls of fixed size.

Recall the following standard definition.

Definition 1.3. The *covering radius* of a set S in $X \subseteq \mathbb{R}^d$ is defined as

$$\inf\left\{r > 0 : \cup_{x\in S} B(x,r) \supseteq X\right\}.$$

Observe then that given a subset $X \subseteq \mathbb{R}^d$, the covering radius of $\mathrm{supp}(Z(t))$ in X is a measure of the density of $\mathrm{supp}(Z(t))$ in X. For finer results on the distribution of mass of Z in $\mathbb{R}^d$, in Theorem 2.1, we first extend (1.2) to linearly moving balls of time-dependent (exponentially decreasing) radius $r = r(t)$. In Theorem 2.4, via a covering by sufficiently many of such smaller balls, we obtain an LD result on the covering radius of the support of BBM in subcritical balls. In Theorem 2.8, building on Theorem 2.4, we obtain large deviation estimates as $t \to \infty$ on the volume of the $r(t)$-enlargement of BBM (see Definition 2.7).

1.2. History and related problems

At the root of the present work is the strong law of large numbers (SLLN) for the local mass of BBM [17, Corollary, p. 222], where Watanabe established an almost sure result on the asymptotic behavior of certain branching Markov processes, which covers the SLLN for local mass of BBM in fixed Borel sets in $\mathbb{R}^d$ as a special case. This was extended by Biggins [2, Corollary 4] to linearly moving Borel sets. The result of Biggins was originally cast in the setting of a branching random walk in discrete time, and extended in the same paper to the continuous setting of a BBM.

We now review various LD results concerning the mass of BBM. First, we consider probabilities of absence or presence. Let $X_{\max}(t)$ denote the position of the rightmost particle at time t of a BBM in $\mathbb{R}$, and for any $d \geq 1$ let

$$M_t := \inf\{r > 0 : \mathrm{supp}(Z(t)) \subseteq B(0,r)\}$$

as before. Set $v = \sqrt{2\beta}$. Recall that by Theorem A, for large t, typically there is mass outside $B(0, rt)$ when $r < v$, but no mass outside $B(0, rt)$ when $r > v$. In [4], the large-time asymptotics of LD probabilities $P(X_{\max}(t) \geq rt)$ for $r > v$ were found when $d = 1$,

*The rate function in Theorem B is related to the one in Theorem 2.1 as $J(\theta, a) = I(\theta, 0, a)$ (see (2.2) and (2.3)).

where $P(X_{\max}(t) \geq rt)$ is a probability of presence in a region where there would typically be no mass. In [7], the asymptotics of $P(M_t \leq rt)$ for $0 < r < v$ were found in any dimension, and note that in this case $P(M_t \leq rt)$ is a probability of absence outside $B(0, rt)$ where there would typically be mass. Recently in [6], the asymptotic behavior to the leading order of $P(X_{\max}(t) \leq rt)$ for $r < v$ was found when $d = 1$, where r was allowed to be negative as well. This result was then refined in [5], where the precise asymptotics of the same lower deviation probability was obtained. More generally, concerning the mass of BBM in time-dependent domains, fewer results are available. In [1], the upper tail asymptotics for the mass inside $[rt, \infty)$, $r < v$ were found for a BBM in $\mathbb{R}$. Due to [2, Corollary 4], the mass inside $[rt, \infty)$ at time t is typically $\exp[\beta(1-\theta^2) + o(t)]$, and in [1], LD probabilities $P(Z_t([rt, \infty)) \geq e^{\beta a t})$ were studied for $1 - \theta^2 < a < 1$.

The current work can be regarded as a sequel to [14] and a prequel to [15] under the common theme of spatial distribution of mass in BBM. In [14], two LD results in the downward direction as $t \to \infty$ were obtained concerning the mass of BBM: the first one, as detailed in Subsection 1.1, was on the mass of BBM falling in linearly moving balls of fixed radius, and the second one was on the mass falling outside linearly expanding balls centered at the origin. In both cases, the asymptotic rate of decay was found for the probability that the mass is atypically small in the respective time-dependent domain. In [15], a branching Brownian sausage with radius exponentially decaying in time was studied, and almost sure limit theorems as $t \to \infty$ on its volume were obtained in all dimensions.

As for the density of BBM, in [9], Grigor'yan and Kelbert established sufficient conditions for the transience and recurrence of a general class of BBMs with time-dependent branching rates and mechanisms on Riemannian manifolds, where the term *recurrence* therein is equivalent to the almost sure density of the full range of BBM in the manifold.

Outline: The rest of the paper is organized as follows. In Section 2, we present our main results. In Section 3, we develop the preparation needed, including the statement and proof of several introductory results, for the proofs of Theorem 2.1 and Theorem 2.4. Section 4 is on the large deviations of the mass of BBM in moving and shrinking balls, including the proof of Theorem 2.1. Section 5 is on the density of BBM in subcritical balls, including the proof of Theorem 2.4. In Section 6, we prove almost sure results on the large-time behavior of $r(t)$-enlargement of the support of BBM when the radius $r(t)$ is exponentially decreasing in t.

2. Results

Our first result is an LD result, giving the large-time asymptotic rate of decay for the probability that the mass of BBM inside a linearly moving and exponentially shrinking ball is atypically small on a logarithmic scale. It is an extension of [14, Thm. 1], where linearly moving balls of fixed size were considered. Here, the radius of the moving ball is time-dependent as well.

Theorem 2.1 (Lower tail asymptotics for mass inside a moving and shrinking ball)**.** *Let $0 \leq \theta < 1$, $0 \leq k < (1-\theta^2)/d$, $r_0 > 0$ and $\boldsymbol{e}$ be a unit vector in $\mathbb{R}^d$. Let $x : \mathbb{R}_+ \to \mathbb{R}_+$ and $r : \mathbb{R}_+ \to \mathbb{R}_+$ be defined by $x(t) = \theta\sqrt{2\beta}t$ and $r(t) = r_0 e^{-\beta k t}$. For $t \geq 0$, define $B_t = B(x(t)\boldsymbol{e}, r(t))$. Then, for $0 \leq a < 1 - \theta^2 - kd$,*

$$\lim_{t\to\infty} \frac{1}{t} \log P\left(Z_t(B_t) < e^{\beta a t}\right) = -\beta \times I(\theta, k, a), \tag{2.1}$$

where

$$I(\theta, k, a) = \inf_{\sigma \in (0, \bar{\sigma}]} \left[\sigma + \frac{\left(\sqrt{(1-\sigma)^2 - (a+kd)(1-\sigma)} - \theta\right)^2}{\sigma}\right], \tag{2.2}$$

and

$$\bar{\sigma} = \bar{\sigma}(\theta, k, a) = 1 - \frac{a + kd}{2} - \sqrt{\left(\frac{a + kd}{2}\right)^2 + \theta^2}. \tag{2.3}$$

Remark 2.2. In terms of the BBM's optimal strategies, the theorem above means that in order to realize the LD event $\left\{Z_t(B_t) < e^{\beta a t}\right\}$ (see the proof of Theorem 2.1): the system suppresses the branching completely, and sends the single particle to a distance of $\sqrt{2\beta}(\sqrt{(1-\hat{\sigma})^2 - (a+kd)(1-\hat{\sigma})} - \theta)t + o(t)$ in the opposite direction of the center of B_t over $[0, \hat{\sigma}t]$, and then behaves 'normally' in the remaining interval $[\hat{\sigma}t, t]$, where $\hat{\sigma}$ denotes the unique minimizer of the optimization problem in (2.2).

The optimization problem in (2.2) is identical to the one in [14, Eq. 4] with the replacement of the parameter a therein by $a + kd$. The following can be shown to hold:

(i) The function to be minimized in (2.2), call f, is strictly convex, and has a unique minimizer on $(0, 1 - a - kd)$. Denote this minimizer by $\hat{\sigma} = \hat{\sigma}(\theta, k, a)$. Then, $\hat{\sigma}$ satisfies $\hat{\sigma} \leq \bar{\sigma}$.

(ii) If we consider f as $f_{\theta,k,a}$, and keep any two of the three parameters θ, k, a fixed, both $\hat{\sigma}$ and $f(\hat{\sigma})$ are strictly decreasing in the remaining parameter over the allowed set of values for that parameter. This is intuitively obvious since it becomes easier to send less than $e^{\beta a t}$ particles to B_t, i.e., the event $\{Z_t(B_t) < e^{\beta a t}\}$ becomes more likely, as either of θ, k, a increases.

For the proofs of (i) and (ii), and more details on the optimization problem in (2.2), we refer the reader to [14, Sect. 5].

Next, we present the main result of this work, which is on the density of BBM in subcritical balls. First, we recall the following standard definition.

Definition 2.3. A set S is said to be δ-*dense* in $X \subseteq \mathbb{R}^d$ for a given $\delta > 0$ if for any x in X, there exists s in S such that $|s - x| < \delta$.

Observe that if S is δ-dense in X, then the covering radius of S in X is at most δ (see Definition 1.3).

Theorem 2.4 (LD on density of BBM). *Let $0 < \theta < 1$, $0 \leq k < (1-\theta^2)/d$, and for $t > 0$ define $\rho_t := \theta\sqrt{2\beta}t$. For $t > 0$ and a function $r : \mathbb{R}_+ \to \mathbb{R}_+$, define the event A_t^r as*

$$A_t^r := \{supp(Z(t)) \text{ is not } r(t)\text{-dense in } B(0, \rho_t)\}.$$

If r is defined by $r(t) = r_0\, e^{-\beta k t}$, where $r_0 > 0$, then

$$\lim_{t\to\infty} \frac{1}{t} \log P(A_t^r) = -\beta \times I(\theta, k, 0). \tag{2.4}$$

Note that the rate constant in (2.4) is a measure of how fast the support of BBM becomes $r(t)$-dense in the linearly expanding ball $B = (B(0, \rho_t))_{t\geq 0}$. Via a Borel-Cantelli argument, Theorem 2.4 leads to the following corollary, which is on the density of the full range of BBM. We provide a proof for completeness.

Corollary 2.5 (Density of BBM). *Let Z be a strictly dyadic BBM with branching rate $\beta > 0$, and let R be its full range as defined in* (1.1). *Then, in any dimension $d \geq 1$, R is dense in $\mathbb{R}^d$ almost surely.*

Proof. For concreteness, set $\theta = 1/\sqrt{2}$ in the definition of ρ_t in the statement of Theorem 2.4 so that $\rho_t = \sqrt{\beta}t$. For $n \in \mathbb{N}$, let F_n be the event that $R(n)$ is not $(1/n)$-dense in $B(0, \rho_n)$. Note that for any k, $1/n \geq e^{-kn}$ for all large n, and for any n, $\text{supp}(Z(n)) \subseteq R(n)$. Therefore, Theorem 2.4 implies that there exist $c > 0$ and $j \in \mathbb{N}$ such that for $n \geq j$, $P(F_n) \leq e^{-cn}$. Since $\sum_{n=j}^{\infty} P(F_n) \leq 1/(1-e^{-c}) < \infty$, by Borel-Cantelli lemma, with probability one, only finitely many F_n occur. This means that $P(\Omega_0) = 1$, where

$$\Omega_0 := \{\omega : \exists\, n_0 = n_0(\omega) \text{ such that } \forall\, n \geq n_0,\ R(n)(\omega) \text{ is } (1/n)\text{-dense in } B(0,\rho_n)\}\,.$$

Let $\omega \in \Omega_0$. Then, there exists $n_0(\omega)$ such that for all $n \geq n_0$, $R(n)(\omega)$ is $(1/n)$-dense in $B(0,\rho_n)$. Let $x \in \mathbb{R}^d$ and $\varepsilon > 0$. Consider $B(x,\varepsilon)$. Choose N large enough so that

$$N > n_0, \quad x \in B(0,\rho_N), \quad \frac{1}{N} < \varepsilon.$$

For instance, choosing $N > \max\{n_0, |x|/\sqrt{\beta}, 1/\varepsilon\}$ suffices. Then, $B(x,\varepsilon) \cap R(N) \neq \emptyset$, which in view of $R(N) \subseteq R$ implies that $B(x,\varepsilon) \cap R \neq \emptyset$. Therefore, $P(R \text{ is dense in } \mathbb{R}^d) \geq P(\Omega_0) = 1$. □

Remark 2.6. We note that Corollary 2.5 is not a new result. Via a similar Borel-Cantelli argument as the one above, one can deduce the almost sure density of the full range of BBM from Watanabe's SLLN [17, Corollary, p. 222] for the local mass of BBM. Also, Corollary 2.5 can be recovered as a special case of [9, Thm. 8.1], which provides sufficient conditions for the transience or recurrence of a general class of branching diffusions on Riemannian manifolds, including the BBM in $\mathbb{R}^d$.

The concept of r-density of $Z(t)$ naturally leads to the following definition.

Definition 2.7 (Enlargement of BBM)**.** Let $Z = (Z(t))_{t\geq 0}$ be a BBM. For $t \geq 0$, we define the *r-enlargement of BBM at time t* corresponding to Z as

$$Z_t^r := \bigcup_{x \in \mathit{supp}(Z(t))} B(x,r).$$

For a function $r : \mathbb{R}_+ \to \mathbb{R}_+$, we may similarly define the *$r(t)$-enlargement of BBM* as $Z_t^{r_t} := \cup_{x \in \mathit{supp}(Z(t))} B(x,r_t)$, where we have set $r_t = r(t)$ for notational convenience. For a Borel set $A \subseteq \mathbb{R}^d$, we say *volume* of A to refer to its Lebesgue measure, which we denote by $\mathsf{vol}(A)$, and use ω_d to denote the volume of the d-dimensional unit ball. The following result may partially (see (2.5)) be viewed as a corollary of Theorem 2.4, and concerns the behavior as $t \to \infty$ of the r_t-enlargement of BBM in $\mathbb{R}^d$ with r_t decaying exponentially as $r_t = r_0\, e^{-\beta k t}$. It says, provided that the decay rate of r_t is not too large, the typical volume of $Z_t^{r_t}$ is $[2\beta(1-kd)]^{d/2}\omega_d t^d + o(t^d)$ and that deviations of order t^d are exponentially unlikely.

Theorem 2.8 (LD on volume of enlargement of BBM)**.** *Let $0 \leq k \leq 1/d$, $r_0 > 0$ and $r : \mathbb{R}_+ \to \mathbb{R}_+$ be defined by $r(t) = r_0\, e^{-\beta k t}$. Then, for every $\varepsilon > 0$ there exist $c_1 > 0$ and $c_2 > 0$ such that for all large t,*

$$P\left(\frac{\mathsf{vol}(Z_t^{r_t})}{t^d} \leq [2\beta(1-kd-\varepsilon)]^{d/2}\omega_d\right) \leq e^{-c_1 t}, \tag{2.5}$$

and

$$P\left(\frac{\mathsf{vol}(Z_t^{r_t})}{t^d} \geq [2\beta(1-kd+\varepsilon)]^{d/2}\omega_d\right) \leq e^{-c_2 t}. \tag{2.6}$$

3. Preparations

Notation: We introduce further notation for the rest of the manuscript. For $x \in \mathbb{R}^d$, we use $|x|$ to denote its Euclidean norm. We use $c, c_0, c_1, \ldots$ as generic positive constants, whose values may change from line to line. If we wish to emphasize the dependence of c on a parameter p, then we write c_p or $c(p)$. We write $o(t)$ to refer to $g(t)$, where $g : \mathbb{R}_+ \to \mathbb{R}_+$ is a generic function satisfying $g(t)/t \to 0$ as $t \to \infty$. We use $a \wedge b$ and $a \vee b$ to denote, respectively, the minimum and maximum of the numbers a and b.

We denote by $X = (X(t))_{t\geq 0}$ a generic standard Brownian motion in d-dimensions, and use $\mathbf{P}_x$ and $\mathbf{E}_x$, respectively, as the law of X started at position $x \in \mathbb{R}^d$, and the corresponding expectation. Also, for $t > 0$, $x, y \in \mathbb{R}^d$, and a Borel set $A \subseteq \mathbb{R}^d$, we denote by $p(t, x, y)$ the Brownian transition density and use $p(t, x, A) := \int_A p(t, x, y)dy$, where dy stands for the Lebesgue measure. Set $p(t, A) := p(t, 0, A)$.

The following result says, the probability that there are no particles of BBM in a ball of fixed radius is an increasing function of the distance between the center of the ball and the starting point of the BBM. This is intuitively obvious, and is a direct consequence of the facts that the Brownian transition density is a decreasing function of $|x - y|$ and that each particle of BBM performs an independent Brownian motion while alive.

Lemma 3.1 (Monotonicity of probability of absence)**.** *Let x_1 and x_2 be in $\mathbb{R}^d$ with $|x_1| > |x_2|$, and $r > 0$ be fixed. Define $B_1 = B(x_1, r)$ and $B_2 = B(x_2, r)$. Then for any $t > 0$,*

$$P\left(Z_t(B_1) = 0\right) \geq P\left(Z_t(B_2) = 0\right).$$

Proof. Fix $r > 0$ and let $g : \mathbb{R}_+ \times \mathbb{R}^d \to [0, 1]$ be defined by

$$g(t, x) = P\left(Z_t(B(x, r)) = 0\right).$$

Let H be any hyperplane in $\mathbb{R}^d$ that does not contain the origin. The hyperplane H splits $\mathbb{R}^d$ into two half-spaces; let S be the half-space containing the origin. Also, let τ be the first branching time of Z, and T be the first hitting time of H by Z. Condition the process on the events $\{\tau > t\}$, $\{T < \tau \leq t\}$, and $\{\tau \leq T \wedge t\}$, which form a partition of the sample space, to obtain

$$\begin{aligned} g(t, x) = e^{-\beta t}[1 - p(t, B(x, r))] + P(Z_t(B(x, r)) = 0 \mid T < \tau \leq t)P(T < \tau \leq t) + \\ \int_0^t \int_S \left[g^2(t - s, x - y)\right] \widetilde{p}(s, 0, y)dy\, \beta e^{-\beta s} ds, \end{aligned}$$

where $\widetilde{p}(s, 0, y)$ is the transition density of Brownian motion conditioned to stay in S up to time s. Now fix $x_1, x_2 \in \mathbb{R}^d$ with $|x_1| > |x_2|$, and let

$$H := \{x \in \mathbb{R}^d : |x_1 - x| = |x_2 - x|\}, \quad S_2 := \{x \in \mathbb{R}^d : |x_1 - x| > |x_2 - x|\}.$$

Observe that by assumption, S_2 is the half-space containing the origin. Then,

$$\begin{aligned} g(t, x_2) - g(t, x_1) = e^{-\beta t}[p(t, B_1) - p(t, B_2)] + \\ P(T < \tau \leq t)\left[P(Z_t(B_2) = 0 \mid T < \tau \leq t) - P(Z_t(B_1) = 0 \mid T < \tau \leq t)\right] + \\ \int_0^t \int_{S_2} \left[g^2(t - s, x_2 - y) - g^2(t - s, x_1 - y)\right] \widetilde{p}(s, 0, y)dy\, \beta e^{-\beta s} ds. \end{aligned} \tag{3.1}$$

The first term on the right-hand side of (3.1) is negative due to the monotonicity of $p(t, x, y)$ in $|x - y|$. Furthermore, since T is the first hitting time of H, the second term on the right-hand side of (3.1) is zero by the strong Markov property of Brownian motion applied at time T, and the spherical symmetry of BBM. Hence, (3.1) leads to

$$\begin{aligned} g(t, x_2) - g(t, x_1) &\leq \int_0^t \int_{S_2} \left[g^2(t - s, x_2 - y) - g^2(t - s, x_1 - y)\right] \widetilde{p}(s, 0, y)dy\, \beta e^{-\beta s} ds \\ &= \int_0^t \int_{S_2} \left[g^2(u, x_2 - y) - g^2(u, x_1 - y)\right] \widetilde{p}(t - u, 0, y)dy\, \beta e^{-\beta(t-u)} du. \end{aligned} \tag{3.2}$$

Define

$$w(t, x) := g(t, x_2 - x) - g(t, x_1 - x), \quad \overline{w} := w \vee 0.$$

Note that

$$g^2(u,x_2-x)-g^2(u,x_1-x)=[g(u,x_2-x)+g(u,x_1-x)][g(u,x_2-x)-g(u,x_1-x)]$$
$$\leq 2\,\overline{w}(u,x).$$

Then, it follows from (3.2) that

$$\overline{w}(t,0)\leq \int_0^t\int_{S_2}\overline{w}(u,y)\widetilde{p}(t-u,0,y)dy\,2\beta du. \tag{3.3}$$

Note that if $\overline{w}(t,0)=0$, then (3.3) holds since the right-hand side is nonnegative, and if $\overline{w}(t,0)>0$, then (3.3) holds by definition of $\overline{w}$ and by (3.2). For $u\geq 0$, define

$$F(u):=\sup_{z\in S_2}\overline{w}(u,z).$$

Then, (3.3) yields

$$\overline{w}(t,0)\leq \int_0^t F(u)\int_{S_2}\widetilde{p}(t-u,0,y)dy\,2\beta du=\int_0^t F(u)2\beta du.$$

Now let $z_0\in S_2$. Recall that we use P_x to denote the probability for Z when the process starts with a single particle at $x\in\mathbb{R}^d$. By translation invariance, we have

$$\begin{aligned} w(t,z_0)&=P\left(Z_t(B(x_2-z_0,r))=0\right)-P\left(Z_t(B(x_1-z_0,r))=0\right)\\ &=P_{z_0}\left(Z_t(B(x_2,r))=0\right)-P_{z_0}\left(Z_t(B(x_1,r))=0\right). \end{aligned}$$

Then, by going through similar steps as (3.1)-(3.3), we obtain

$$\overline{w}(t,z_0)\leq \int_0^t\int_{S_2}\overline{w}(u,y)\widetilde{p}(t-u,z_0,y)dy\,2\beta du\leq \int_0^t F(u)2\beta du.$$

This implies that

$$\sup_{z\in S_2}\overline{w}(t,z)=F(t)\leq \int_0^t F(u)2\beta du.$$

Then, by Grönwall's inequality we conclude that $F(t)\leq 0$. This implies that $\overline{w}(t,0)\leq 0$. But, $\overline{w}(t,0)\geq 0$ by definition. Therefore, $\overline{w}(t,0)=0$, that is, $g(t,x_2)-g(t,x_1)\leq 0$, which means that $g(t,x_1)\geq g(t,x_2)$ as claimed. □

Next, we list two well-known results; the first one is about the global growth of branching systems, and the second one about the large-time asymptotic probability of atypically large Brownian displacements. These results will be useful in the proofs of the main theorems. For the proofs of Proposition A and Proposition B, see for example [10, Sect. 8.11] and [16, Lemma 5], respectively.

Proposition A (Distribution of mass in branching systems)**.** For a strictly dyadic continuous-time branching process $N=(N_t)_{t\geq 0}$ with constant branching rate $\beta>0$, the probability distribution at time t is given by

$$P(N_t=k)=e^{-\beta t}(1-e^{-\beta t})^{k-1},\quad k\geq 1,$$

from which it follows that

$$P(N_t>k)=(1-e^{-\beta t})^k. \tag{3.4}$$

Proposition B (Linear Brownian displacements)**.** Let $X=(X(t))_{t\geq 0}$ represent a standard d-dimensional Brownian motion starting at the origin, and $\mathbf{P}_0$ the corresponding probability. Then, for $\gamma>0$ as $t\to\infty$,

$$\mathbf{P}_0\left(\sup_{0\leq s\leq t}|X(s)|>\gamma t\right)=\exp[-\gamma^2 t/2+o(t)].$$

4. Mass in a moving and shrinking ball

We open this section with some terminology that are used often in the proofs below.

Definition 4.1 (SES)**.** A generic function $g : \mathbb{R}_+ \to \mathbb{R}_+$ is called *super-exponentially small (SES)* if $\lim_{t\to\infty} \log g(t)/t = -\infty$.

Definition 4.2 (Overwhelming probability)**.** Let $(A_t)_{t>0}$ be a family of events. We say that A_t occurs *with overwhelming probability* as $t \to \infty$ if there are a constant $c > 0$ and time t_0 such that

$$P(A_t^c) \le e^{-ct} \quad \text{for all} \quad t \ge t_0,$$

where A^c denotes the complement of event A.

Also, for a function $g : \mathbb{R}_+ \to \mathbb{R}_+$, we will often use $g_t = g(t)$ for notational convenience. The following lemma says that exponentially few particles in a moving and shrinking ball, is exponentially unlikely. It constitutes the first step of a two-step bootstrap argument, which we use to prove the upper bound of (2.1) in Theorem 2.1. The proof of the upper bound of Theorem 2.1 will sharpen the constant on the right-hand side of (4.1) below.

Lemma 4.3. *Let $0 \le \theta < 1$, $0 \le k < (1-\theta^2)/d$, $r_0 > 0$, and $\boldsymbol{e}$ be a unit vector in $\mathbb{R}^d$. Let $x : \mathbb{R}_+ \to \mathbb{R}_+$ and $r : \mathbb{R}_+ \to \mathbb{R}_+$ be defined by $x(t) = \theta\sqrt{2\beta}t$ and $r(t) = r_0 e^{-\beta k t}$. For $t \ge 0$, define $B_t = B(x(t)\boldsymbol{e}, r(t))$. Then, for each $0 \le a < 1 - \theta^2 - kd$, there exists a constant $c = c(\beta, d, \theta, k, a) > 0$ such that*

$$\limsup_{t\to\infty} \frac{1}{t} \log P\left(Z_t(B_t) < e^{\beta a t}\right) \le -c. \tag{4.1}$$

Remark 4.4. Note that $B = (B_t)_{t\ge 0}$ represents a linearly moving and exponentially shrinking ball. Using a many-to-one formula, we have

$$\begin{aligned} E[Z_t(B_t)] = E[Z_t(\mathbb{R}^d)] \times p(t, B_t) &= e^{\beta t} \times \frac{1}{(2\pi t)^{d/2}} \int_{B_t} e^{-|x|^2/(2t)} \mathrm{d}x \\ &= e^{\beta t(1-\theta^2-kd)+o(t)}, \end{aligned} \tag{4.2}$$

where $p(t, A)$ is as before the Brownian transition probability from the origin to the Borel set A at time t. As reflected by (4.2), the growth rate of $Z_t(B_t)$ consists of three pieces: the first term in the exponent on the right-hand side of (4.2) contributes positively and is simply the growth rate of the total mass of BBM, the second and third terms contribute negatively to the exponent, and come from a ‘one-particle picture,’ where a Brownian particle has linear displacement and falls inside a specified ball of exponentially decaying radius. Since $a < 1 - \theta^2 - kd$ in the lemma above, a is an atypically small exponent for the mass in B_t at time t.

Proof. To start the proof, for $0 \le a < 1 - \theta^2 - kd$ and $t > 0$, let

$$A_t := \left\{Z_t(B_t) < e^{\beta a t}\right\},$$

and choose $0 < \delta < 1$ small enough so as to satisfy

$$a < 1 - \theta^2 - kd - \delta.$$

Consider the ball $B(x_t\mathbf{e}, r_0)$ so that $B_t \subseteq B(x_t\mathbf{e}, r_0)$ for all $t > 0$. Next, for $t > 0$, define the event

$$E_t := \left\{Z_{t-1}\left(B(x_t\mathbf{e}, r_0)\right) \ge \exp\left[\beta(1-\theta^2-\delta)t\right]\right\},$$

and estimate

$$P(A_t) \le P(A_t \mid E_t) + P(E_t^c). \tag{4.3}$$

Using Theorem B, since $\beta(1 - \theta^2 - \delta)$ is an atypically small exponent for the mass inside $B(x_t\mathbf{e}, r_0)$ at time $t - 1$, for all large t, $P(E_t^c)$ can be bounded from above as

$$P(E_t^c) \le e^{-c_1 t} \tag{4.4}$$

for some $c_1 = c_1(\beta, \theta, \delta) > 0$. (Note that $\delta = \delta(d, \theta, k, a)$.) Next, we show that $P(A_t \mid E_t)$ on the right-hand side of (4.3) is SES in t.

Conditional on the event E_t, there are at least $\exp\left[\beta(1-\theta^2-\delta)t\right]$ particles in $B(x_t\mathbf{e}, r_0)$ at time $t-1$. Apply the branching Markov property at time $t-1$. For a lower bound on the mass inside B_t at time t, neglect possible branching of the particles present in $B(x_t\mathbf{e}, r_0)$ at time $t-1$ over the period $[t-1, t]$, and suppose that each one evolves as an independent Brownian particle starting from her position at time $t-1$. A standard calculation yields, uniformly over $x \in B(0, r_0)$,

$$\begin{aligned} p(1, x, B(0, r_t)) &= \int_{B(0,r_t)} p(1, x, y)dy = \frac{1}{(\sqrt{2\pi})^d} \int_{B(0,r_t)} e^{-|y-x|^2/2} dy \\ &\geq \frac{e^{-(r_0+r_t)^2/2}}{(\sqrt{2\pi})^d} \,\mathsf{vol}\,(B(0, r_t)) \\ &\geq \frac{e^{-(2r_0)^2/2}}{(\sqrt{2\pi})^d}\, \omega_d r_t^d \; = \; c_2 \exp[-\beta(kd)t] \end{aligned}$$

for some constant $c_2 > 0$, and we have used in the second inequality that $r_t \leq r_0$ for all $t > 0$. By translation invariance, uniformly over $x \in B(x_t\mathbf{e}, r_0)$,

$$p(1, x, B(x_t\mathbf{e}, r_t)) \geq c_2 \exp[-\beta(kd)t].$$

Now for $t > t_0$, where t_0 is large enough, let

$$p_t := c_2 e^{-\beta(kd)t}, \;\; M_t := \left\lceil e^{\beta(1-\theta^2-\delta)t} \right\rceil,$$

and let Y_t be a random variable, which under the law Q, has a binomial distribution with parameters M_t and p_t. (Here, p_t is the probability of 'success,' and M_t is the number of trials.) Note that each particle in $B(x_t\mathbf{e}, r_0)$ at time $t-1$ moves independently of others over $[t-1, t]$, and that conditional on E_t there are at least M_t particles in $B(x_t\mathbf{e}, r_0)$ at time $t-1$. Therefore, it follows that

$$\begin{aligned} P(A_t \mid E_t) &= P\left(Z_t(B(x_t\mathbf{e}, r_t)) < e^{\beta at} \mid Z_{t-1}(B(x_t\mathbf{e}, r_0)) \geq M_t\right) \\ &= E\left[P_{Z_{t-1}}\left(Z_1(B(x_t\mathbf{e}, r_t)) < e^{\beta at}\right) \mid Z_{t-1}(B(x_t\mathbf{e}, r_0)) \geq M_t\right] \\ &\leq \sup_{(y_1,\ldots,y_{M_t}) \in \prod_{i=1}^{M_t} B(x_t\mathbf{e}, r_0)} P_{\sum_{i=1}^{M_t} \delta_{y_i}} \left(Z_1(B(x_t\mathbf{e}, r_t)) < e^{\beta at}\right) \\ &\leq Q(Y_t \leq e^{\beta at}), \end{aligned} \tag{4.5}$$

where we use $\prod_{i=1}^n A_i$ to denote the Cartesian product of sets $(A_i)_{1\leq i\leq n}$, use P_μ for the law of a BBM starting with the discrete measure $\mu = \sum_{i=1}^n \delta_{x_i}$, and associate Z_t with the discrete measure $\sum_{i=1}^{N_t} \delta_{Z_t^i}$ with Z_t^i denoting the position of the ith particle alive at time t. We bound $Q(Y_t \leq e^{\beta at})$ from above via a standard Chernoff bound as

$$\begin{aligned} Q(Y_t \leq e^{\beta at}) &\leq e^{-p_t M_t} \left(\frac{e p_t M_t}{e^{\beta at}}\right)^{e^{\beta at}} \\ &= \exp\left[-c_2 e^{-\beta(kd)t} \left\lceil e^{\beta t(1-\theta^2-\delta)} \right\rceil\right] \left(\frac{e c_2 e^{-\beta(kd)t} \left\lceil e^{\beta t(1-\theta^2-\delta)} \right\rceil}{e^{\beta at}}\right)^{e^{\beta at}}. \end{aligned} \tag{4.6}$$

It follows from (4.6) that for all large t,

$$Q(Y_t \leq e^{\beta at}) \leq \exp\left[-c_2 e^{\beta t(1-\theta^2-kd-\delta)}\right] \exp\left(\beta t e^{\beta at}\right),$$

which is SES in t since $a < 1 - \theta^2 - kd - \delta$ by the choice of δ. Therefore, it follows from (4.5) that $P(A_t \mid E_t)$ is SES in t as well. This completes the proof in view of (4.3) and (4.4). $\square$

4.1. Proof of Theorem 2.1

Theorem 2.1 is proved in the same spirit as [14, Thm. 1]. For the lower bound, we find a strategy that realizes the desired event with optimal probability on a logarithmic scale. The proof of the upper bound can be viewed as the second step of a bootstrap argument, whose first step was completed by Lemma 4.3; it sharpens the constant on the right-hand side of (4.1) so as to show that the strategy that gives the lower bound is indeed optimal.

4.1.1. Proof of the lower bound. Fix $0 \le \theta < 1$, $0 \le k < (1-\theta^2)/d$, and a unit vector $\mathbf{e}$. For $0 \le a < 1-\theta^2 - kd$ and $t \ge 0$, define the event

$$A_t = \left\{ Z_t(B_t) < e^{\beta a t} \right\},$$

and define

$$\bar\sigma = \bar\sigma(\theta,k,a) = 1-(a+kd)/2 - \sqrt{((a+kd)/2)^2+\theta^2},$$

which is chosen so that $(1-\bar\sigma)^2 - (a+kd)(1-\bar\sigma) = \theta^2$. Let $0<\sigma\le\bar\sigma$ and $\varepsilon>0$. Let F_t be the event that in the time interval $[0,\sigma t]$, the branching is completely suppressed and the initial Brownian particle is moved to a point whose coordinate in the direction of $\mathbf{e}$ is less than $-d(t)$, where

$$d(t) := \left(\sqrt{(1-\sigma)^2-(a+kd)(1-\sigma)} - \theta + \varepsilon\right)\sqrt{2\beta}t.$$

That is,

$$F_t = \left\{ N_{\sigma t} = 1,\ \langle X_{\sigma t}, \mathbf{e}\rangle < -d(t) \right\},$$

where $X_{\sigma t}$ is the position of the initial particle at time σt and $\langle\cdot,\cdot\rangle$ is the standard inner product in $\mathbb{R}^d$. Then, since $\langle X_\cdot, \mathbf{e}\rangle$ is equal in law to a one-dimensional Brownian motion, by using the Brownian transition density $p(t,0,x) = (2\pi t)^{-1/2} e^{-x^2/(2t)}$ in $d=1$ to find the asymptotic behavior as $t\to\infty$ of $p(\sigma t, 0, (-\infty, -d(t))) = (2\pi\sigma t)^{-1/2}\int_{-\infty}^{-d(t)} e^{-x^2/(2\sigma t)}dx$, and using the independence of branching and motion mechanisms of BBM, this partial strategy over $[0,\sigma t]$ has probability

$$P(F_t) = \exp\left[-\beta\left(\sigma + \frac{\left(\sqrt{(1-\sigma)^2-(a+kd)(1-\sigma)} - \theta + \varepsilon\right)^2}{\sigma}\right)t + o(t)\right], \tag{4.7}$$

where the first term under the exponent comes from suppressing the branching, and the second term from the linear Brownian displacement. By the Markov property applied at time σt, it is clear that $P(A_t \mid F_t)$ is the same as the probability that a BBM starting with a single particle at position $X_{\sigma t}$ contributes a mass of less than $e^{\beta a t}$ to B_t at time $(1-\sigma)t$. Since the distance between $X_{\sigma t}$ and the center of B_t is at least

$$d(t) + \theta\sqrt{2\beta}t = \left(\sqrt{(1-\sigma)^2-(a+kd)(1-\sigma)} + \varepsilon\right)\sqrt{2\beta}t$$

conditional on the event F_t, the Markov inequality yields

$$\begin{aligned} P(A_t^c \mid F_t) &= P\left(Z_t(B_t) \ge e^{\beta a t} \mid F_t\right) \le \frac{E[Z_t(B_t)\mid F_t]}{e^{\beta a t}} \\ &\le \frac{\exp\left[\left(\beta(1-\sigma) - \frac{1}{2}\frac{\left(\sqrt{(1-\sigma)^2-(a+kd)(1-\sigma)}+\varepsilon\right)^2(\sqrt{2\beta})^2}{1-\sigma} - \beta kd\right)t + o(t)\right]}{e^{\beta a t}}, \end{aligned} \tag{4.8}$$

where we have used the Markov property and (4.2) in the last inequality. Now since

$$\beta(1-\sigma) - \frac{1}{2}\frac{\left(\sqrt{(1-\sigma)^2-(a+kd)(1-\sigma)}+\varepsilon\right)^2(\sqrt{2\beta})^2}{1-\sigma} - \beta kd < \beta a,$$

(4.8) implies that $P(A_t \mid F_t) = \exp[o(t)]$. Then, from the estimate $P(A_t) \geq P(F_t)P(A_t \mid F_t)$ and (4.7), it follows that

$$\liminf_{t\to\infty} \frac{1}{t} \log P(A_t) \geq -\beta \left[\sigma + \frac{\left(\sqrt{(1-\sigma)^2 - (a+kd)(1-\sigma)} - \theta + \varepsilon\right)^2}{\sigma} \right]. \tag{4.9}$$

Let $\varepsilon \to 0$, and optimize the right-hand side of (4.9) over $\sigma \in (0, \bar{\sigma}]$ to complete the proof of the lower bound.

4.1.2. Proof of the upper bound. We refer the reader to the proof of the upper bound of [14, Thm. 1]; simply change the parameter a by $a + kd$ in the equations (19), (21), (24)-(26) therein. The proof of the upper bound of Theorem 2.1 is otherwise identical to that of [14, Thm. 1]. We note that in the present work a similar (but not identical) technique is used later for the proof of the upper bound of Theorem 2.4. Therefore, to avoid duplication, here we simply refer the reader to the proof of [14, Thm. 1].

5. Density of BBM

In this section, we prove Theorem 2.4. The lower bound is a direct consequence of Theorem 2.1. The proof of the upper bound uses a method similar to that of [8, Thm. 1] and [14, Thm. 1], along with some geometric arguments. In the proofs below, $0 < \theta < 1$, $0 \leq k < (1-\theta^2)/d$ and $r_0 > 0$ are fixed, and $r_t = r_0 \, e^{-\beta k t}$ for $t \geq 0$.

5.1. Theorem 2.4 – Proof of the lower bound

Let $0 < \theta' < \theta$. Then, $0 \leq k < (1-\theta'^2)/d$ as well. For $t \geq 0$, let $B_t := B(x_t \mathbf{e}, r_t)$, where $x_t = \theta' \sqrt{2\beta} t$, and $\mathbf{e} = (1, 0, \ldots, 0)$ is the unit vector in the direction of the first coordinate. Then, by Theorem 2.1,

$$P(Z_t(B_t) = 0) = \exp\left[-\beta \, I(\theta', k, 0) + o(t)\right].$$

Since $\{Z_t(B_t) = 0\} \subseteq A_t^r = \{\text{supp}(Z(t)) \text{ is not } r_t\text{-dense in } B(0, \theta\sqrt{2\beta}t)\}$ for all large t, it follows that

$$\liminf_{t\to\infty} \frac{1}{t} \log P\left(A_t^r\right) \geq -\beta I(\theta', k, 0).$$

Let $\theta' \to \theta$ and use the continuity of $I(\theta', k, 0)$ in θ' to complete the proof.

5.2. Theorem 2.4 – Proof of the upper bound

Throughout this proof, we use

$$B_t := B(\theta\sqrt{2\beta}t\mathbf{e}, r_0), \quad \mathcal{B}_t := B(0, \theta\sqrt{2\beta}t).$$

The proof is broken into three parts for better readability. The first two parts are on the r_t-density of BBM only within B_t. The last part extends the r_t-density of BBM to the entire subcritical ball $\mathcal{B}_t$. The first part is a suitable modification of the argument that was used to prove [14, Thm. 1], whereas the other two parts are based on Euclidean geometry. In the rest of the proof, fix the dimension d, and let

$$n_t := 2^d \left\lceil \sqrt{d} \, e^{\beta k t} \right\rceil^d. \tag{5.1}$$

5.2.1. Part I: Any n_t-collection of balls within B_t. Let $(x_j : 1 \le j \le n_t)$ be any collection of n_t points in B_t, where we suppress the t-dependence of x_j for ease of notation. For each j, define

$$B_t^j := B(x_j, r_t/(2\sqrt{d}))$$

so that each B_t^j is a ball with center lying in B_t. For $t > 0$ and $1 \le j \le n_t$, define the events

$$A_t^j := \{Z_t(B_t^j) = 0\}, \quad A_t := \bigcup_{1 \le j \le n_t} A_t^j.$$

Observe that A_t is the event that at least one B_t^j is empty at time t. Recall that $N_t = Z_t(\mathbb{R}^d)$, and for $t > 1$ define the random variable

$$\phi_t = \sup\{\sigma \in [0,1] : N_{\sigma t} \le \lfloor t \rfloor\}.$$

Observe that for $x \in [0,1]$, we have $\{\phi_t \ge x\} \subseteq \{N_{xt} \le \lfloor t \rfloor + 1\}$. We start by conditioning on ϕ_t. Recall the definition of $\bar{\sigma}$ from (2.3), and set

$$\bar{\sigma} := \bar{\sigma}(\theta, k, 0) = 1 - (kd)/2 - \sqrt{(kd/2)^2 + \theta^2}. \tag{5.2}$$

Note that $\bar{\sigma} > 0$ since $kd < 1 - \theta^2$. Choose $n_0 \in \mathbb{N}$ large enough so that $\lfloor \bar{\sigma} n_0 - 1 \rfloor - 1 \ge 0$. Then, for every $n \ge n_0$,

$$\begin{aligned} P(A_t) &= \sum_{i=0}^{\lfloor \bar{\sigma} n - 1 \rfloor - 1} P\left(A_t \cap \left\{\frac{i}{n} \le \phi_t < \frac{i+1}{n}\right\}\right) + P\left(A_t \cap \left\{\phi_t \ge \frac{\lfloor \bar{\sigma} n - 1 \rfloor}{n}\right\}\right) \\ &\le \sum_{i=0}^{\lfloor \bar{\sigma} n - 1 \rfloor - 1} \exp\left[-\beta \frac{i}{n} t + o(t)\right] P_t^{(i,n)}(A_t) + \exp\left[-\beta\left(\bar{\sigma} - \frac{2}{n}\right) t + o(t)\right], \end{aligned} \tag{5.3}$$

where we use (3.4), which implies $P(N_{(i/n)t} \le \lfloor t \rfloor + 1) = \exp[-\beta(i/n)t + o(t)]$, to control $P(\frac{i}{n} \le \phi_t < \frac{i+1}{n})$, and introduce the conditional probabilities

$$P_t^{(i,n)}(\cdot) = P\left(\cdot \;\middle|\; \frac{i}{n} \le \phi_t < \frac{i+1}{n}\right), \quad i = 0, 1, \ldots, \lfloor \bar{\sigma} n - 1 \rfloor - 1.$$

For each pair (i,n), where $n \ge n_0$ and $i = 0, 1, \ldots, \lfloor \bar{\sigma} n - 1 \rfloor - 1$, define the interval

$$I^{(i,n)} := [i/n, (i+1)/n),$$

and the radius

$$R_t^{(i,n)} := \sqrt{2\beta}\left(\sqrt{\left(1 - \frac{i+1}{n}\right)^2 - kd\left(1 - \frac{i+1}{n}\right)} - \theta - \varepsilon\right) t, \tag{5.4}$$

where $\varepsilon = \varepsilon(n) > 0$ is chosen small enough so that (5.4) is positive for each $i = 0, 1, \ldots, \lfloor \bar{\sigma} n - 1 \rfloor - 1$.† By definition of ϕ_t, conditional on $\{\phi_t \in I^{(i,n)}\}$, there exists an instant in $[ti/n, t(i+1)/n)$, namely $\phi_t t$, at which there are exactly $\lfloor t \rfloor + 1$ particles in the system. Let $E_t^{(i,n)}$ be the event that among the $\lfloor t \rfloor + 1$ particles alive at $\phi_t t$, there is at least one outside $B_t^{(i,n)} := B\left(0, R_t^{(i,n)}\right)$. Estimate

$$P_t^{(i,n)}(A_t) \le P_t^{(i,n)}\left(E_t^{(i,n)}\right) + P_t^{(i,n)}\left(A_t \mid [E_t^{(i,n)}]^c\right). \tag{5.5}$$

†Indeed, $(1-(i+1)/n)^2 - kd(1-(i+1)/n) \ge (1-\bar{\sigma}+(1/n))^2 - kd(1-\bar{\sigma}+(1/n))$, where the inequality comes from the 'worst case,' where $i = \lfloor \bar{\sigma} n - 1 \rfloor - 1$. On the other hand, observe that $(1-\bar{\sigma})^2 - kd(1-\bar{\sigma}) = \theta^2$ by the choice of $\bar{\sigma}$ (see (5.2)).

Since the ancestral line of each particle is identically distributed as a Brownian trajectory, by the independence of branching and motion mechanisms, Proposition B and the union bound, we have

$$P_t^{(i,n)}\left(E_t^{(i,n)}\right) \leq (\lfloor t\rfloor+1)\exp\left[-\frac{\left(R_t^{(i,n)}\right)^2}{2t(i+1)/n}+o(t)\right]=\exp\left[-\frac{\left(R_t^{(i,n)}\right)^2}{2t(i+1)/n}+o(t)\right]. \quad (5.6)$$

Now consider the second term on the right-hand side of (5.5). Recall that $A_t^j=\{Z_t(B_t^j)=0\}$ and $A_t=\bigcup_{1\leq j\leq n_t}A_t^j$, and the center of each $B_t^j=B(x_j,r_t/(2\sqrt{d}))$ lies in B_t. The union bound gives

$$P_t^{(i,n)}\left(A_t \mid [E_t^{(i,n)}]^c\right)\leq\sum_{j=1}^{n_t}P_t^{(i,n)}\left(A_t^j \mid [E_t^{(i,n)}]^c\right). \quad (5.7)$$

We now find a uniform bound on $P_t^{(i,n)}(A_t^j \mid [E_t^{(i,n)}]^c)$ over $1\leq j\leq n_t$. Observe that

$$\begin{aligned}d_t^{(i,n)} &:= \max_{x\in B_t,\, y\in B_t^{(i,n)}}|x-y|=\theta\sqrt{2\beta}t+r_0+R_t^{(i,n)}\\ &=\sqrt{2\beta}\left(\sqrt{(1-(i+1)/n)^2-kd(1-(i+1)/n)}-\varepsilon\right)t+r_0.\end{aligned} \quad (5.8)$$

Let $p_t^{(y,j)}$ be the probability that a BBM starting with a single particle at $y\in\mathbb{R}^d$ contributes no particles to B_t^j at time t. Recall that by Lemma 3.1, the probability of absence in a ball is monotone increasing in the distance between the center of the ball and the starting point of the BBM. Hence, since $B_t^j=B(x_j,r_t/(2\sqrt{d}))$ with $x_j\in B_t$ for each j, Lemma 3.1 implies that uniformly over $1\leq j\leq n_t$ and $y\in B_t^{(i,n)}$,

$$p_{t(1-(i+1)/n)}^{(y,j)}\leq P\left(Z_{t(1-(i+1)/n)}\left(B(d_t^{(i,n)}\mathbf{e},r_t/(2\sqrt{d}))\right)=0\right), \quad (5.9)$$

where $\mathbf{e}$ denotes any unit vector in $\mathbb{R}^d$. We now estimate the probability on the right-hand side of (5.9). Due to (5.8), the mass inside $B(d_t^{(i,n)}\mathbf{e},r_t/(2\sqrt{d}))$ at time $t(1-(i+1)/n)$ is typically (see the remark following Lemma 4.3)

$$\exp\left[\beta\left(\left(1-\frac{i+1}{n}\right)-\frac{\left(\sqrt{(1-(i+1)/n)^2-kd(1-(i+1)/n)}-\varepsilon\right)^2}{1-\frac{i+1}{n}}-kd\right)t+o(t)\right]. \quad (5.10)$$

Observe that the exponent in (5.10) is positive, which means 0 is an atypically small exponent for the mass considered above. Hence, with the choice of $a=0$ and by substituting $t(1-(i+1)/n)$ for t, Lemma 4.3 implies that there exist $c>0$ and t_0 such that

$$P\left(Z_{t(1-(i+1)/n)}\left(B(d_t^{(i,n)}\mathbf{e},r_t/(2\sqrt{d}))\right)=0\right)\leq e^{-ct}\quad\text{for all}\quad t\geq t_0. \quad (5.11)$$

Note that in applying Lemma 4.3 above, the parameters θ and k in Lemma 4.3 were taken as

$$\frac{\sqrt{(1-(i+1)/n)^2-kd(1-(i+1)/n)}-\varepsilon}{1-(i+1)/n}\quad\text{and}\quad\frac{k}{1-(i+1)/n},$$

respectively. Now, by (5.11), for $t\geq t_0$ and $s>t$,

$$P\left(Z_{s(1-(i+1)/n)}\left(B(d_s^{(i,n)}\mathbf{e},r_s/(2\sqrt{d}))\right)=0\right)\leq e^{-cs}. \quad (5.12)$$

Then, since $d_s^{(i,n)}>d_t^{(i,n)}$ and $r_s<r_t$, using Lemma 1, and then the containment

$$B(d_t^{(i,n)}\mathbf{e},r_s/(2\sqrt{d}))\subseteq B(d_t^{(i,n)}\mathbf{e},r_t/(2\sqrt{d})),$$

we may continue (5.12) with

$$P\left(Z_{s(1-(i+1)/n)}\left(B(d_t^{(i,n)}\mathbf{e}, r_t/(2\sqrt{d}))\right)=0\right)\le e^{-ct}. \tag{5.13}$$

By the strong Markov property applied at time $\phi_t t$ and the independence of particles present at that time, each particle alive at time $\phi_t t$ initiates a BBM in its own right, independently of others. Moreover, recall that $t(1-\phi_t)\ge t(1-(i+1)/n)$ conditional on the event $\phi_t\in I^{(i,n)}$, and there are $\lfloor t\rfloor+1$ particles in $B_t^{(i,n)}$ at time $\phi_t t$ conditional on the event $[E_t^{(i,n)}]^c$. Then, applying the strong Markov property at time $\phi_t t$, (5.9), (5.11) and (5.13) imply that for all large t and $1\le j\le n_t$,

$$P_t^{(i,n)}\left(A_t^j\mid [E_t^{(i,n)}]^c\right)\le \left(e^{-ct}\right)^{\lfloor t\rfloor+1}\le e^{-ct^2}, \tag{5.14}$$

which is SES in t. It follows from (5.7) and (5.14) that

$$P_t^{(i,n)}\left(A_t\mid [E_t^{(i,n)}]^c\right)\le n_t\, e^{-ct^2}=\left\lceil 2\sqrt{d}\,e^{\beta kt}\right\rceil^d e^{-ct^2}=e^{-ct^2+o(t^2)}. \tag{5.15}$$

This means, given $[E_t^{(i,n)}]^c$, it is extremely likely (to the extent that the complement event has SES probability in t) that for each $1\le j\le n_t$ there is at least one particle in $B_t^{(i,n)}$ at time $\phi_t t$ such that the sub-BBM it initiates contributes a particle to B_t^j at time t. From (5.4), (5.5), (5.6) and (5.15), we obtain

$$P_t^{(i,n)}(A_t)\le \exp\left[-\beta\,\frac{\left(\sqrt{(1-\frac{i+1}{n})^2-kd(1-\frac{i+1}{n})}-\theta-\varepsilon\right)^2}{(i+1)/n}\,t+o(t)\right]+\exp\left[-ct^2+o\left(t^2\right)\right]. \tag{5.16}$$

Substituting (5.16) into (5.3), and optimizing over $i\in\{0,1,\dots,\lfloor\bar\sigma n-1\rfloor-1\}$ gives

$$\begin{aligned}&\limsup_{t\to\infty}\frac{1}{t}\log P(A_t)\le\\ &-\beta\left[\min_{i\in\{0,1,\dots,\lfloor\bar\sigma n-1\rfloor-1\}}\left\{\frac{i}{n}+\frac{\left(\sqrt{\left(1-\frac{i+1}{n}\right)^2-kd\left(1-\frac{i+1}{n}\right)}-\theta-\varepsilon\right)^2}{(i+1)/n}\right\}\wedge\left(\bar\sigma-\frac{2}{n}\right)\right],\end{aligned} \tag{5.17}$$

where we use $a\wedge b$ to denote the minimum of a and b. Now first let $\varepsilon\to 0$, and set $\sigma=(i+1)/n$ to obtain

$$\limsup_{t\to\infty}\frac{1}{t}\log P(A_t)\le -\beta\min_{\sigma\in\left\{\frac{1}{n},\frac{2}{n},\dots,\frac{\lfloor\bar\sigma n-1\rfloor}{n}\right\}}\left[\sigma+\frac{(\sqrt{(1-\sigma)^2-kd(1-\sigma)}-\theta)^2}{\sigma}-\frac{1}{n}\right].$$

Then, let $n\to\infty$ to expand the set over which the minimum is taken to $(0,\bar\sigma)\cap\mathbb{Q}$, and use the continuity of the functional form from which the minimum is taken to obtain

$$\limsup_{t\to\infty}\frac{1}{t}\log P(A_t)\le -\beta\inf_{\sigma\in(0,\bar\sigma]}\left[\sigma+\frac{(\sqrt{(1-\sigma)^2-kd(1-\sigma)}-\theta)^2}{\sigma}\right]=-\beta I(\theta,k,0). \tag{5.18}$$

(Note that we have not written the last term on the right-hand side of (5.17) explicitly in (5.18), because once $n\to\infty$, this term becomes $\bar\sigma$, which is attained by the function inside the infimum on the right-hand side of (5.18) if we set $\sigma=\bar\sigma$.)

Remark 5.1. We note that applying a rough union bound on $P\left(\cup_{1\le j\le n_t} A_t^j\right)$ naively along with Theorem 2.1 gives

$$P(A_t) = P\left(\cup_{1\le j\le n_t} A_t^j\right) \le n_t \max_{1\le j\le n_t} P(A_t^j) = \exp[-\beta t(I(\theta,k,0)-kd)+o(t)],$$

which is not the desired upper bound. Indeed, this is not surprising, because the optimal strategy to keep a small ball B_t^j empty at time t is the same for all small balls (so, it is a strategy that works jointly for all B_t^j, $1\le j\le n_t$) contained in the ball of fixed radius r_0. That is, no branching and moving at a linear distance from the origin in the first interval $[0,\sigma t]$.

5.2.2. Part II: Choosing the n_t-collection of balls within B_t. We now choose the collection of points $(x_j : 1\le j\le n_t)$ in $B_t = B(\theta\sqrt{2\beta}t\mathbf{e}, r_0)$ in a useful way. Let $C(0,r_0)$ be the cube centered at the origin with side length $2r_0$ so that $B(0,r_0)$ is inscribed in $C(0,r_0)$. Recall that for $t>0$,

$$n_t := 2^d\left\lceil\sqrt{d}\,e^{\beta kt}\right\rceil^d,\quad r_t = r_0e^{-\beta kt}.$$

Consider the simple cubic packing of $C(0,r_0)$ with balls of radius $r_t/(2\sqrt{d})$. That is, for $m\in\left[-\left\lceil\sqrt{d}e^{\beta kt}\right\rceil,\left\lceil\sqrt{d}e^{\beta kt}\right\rceil-1\right]^d\cap\mathbb{Z}^d =: \mathbf{S}$, introduce the cubes

$$C_m := \left\{z\in\mathbb{R}^d : m_i\frac{r_t}{\sqrt{d}}\le z_i\le(m_i+1)\frac{r_t}{\sqrt{d}}\right\},$$

where we use $x=(x_1,\dots,x_d)\in\mathbb{R}^d$ to denote the coordinates of a point, and inscribe a ball of radius $r_t/(2\sqrt{d})$ inside each of these cubes. Then,

$$\bigcup_{m\in\mathbf{S}} C_m \supset C(0,r_0),$$

and hence $|\mathbf{S}| = n_t$ cubes are sufficient to completely pack $C(0,r_0)$, say with centers $(y_j : 1\le j\le n_t)$. For each j, let $\widehat{B}_j = B(y_j, r_t/(2\sqrt{d}))$. (We suppress the t-dependence in y_j and $\widehat{B}_j$ for ease of notation.) Now consider a simple cubic packing of $\mathbb{R}^d$ by balls $(\mathcal{B}_j : j\in\mathbb{Z}_+)$ of radius $R>0$. That is, for $n\in\mathbb{Z}^d$ and any fixed $y\in\mathbb{R}^d$, introduce the cubes

$$C_n := \left\{z\in\mathbb{R}^d : n_i(2R)\le z_i-y_i\le(n_i+1)(2R)\right\},$$

so that $\cup_{n\in\mathbb{Z}^d} C_n = \mathbb{R}^d$, and inscribe a ball of radius R inside each of these cubes. Let $x\in\mathbb{R}^d$ be any point. Then,

$$\min_j\max_{z\in\mathcal{B}_j}|x-z| < \max_{x,y\in[0,2R]^d}|x-y| = 2R\max_{x,y\in[0,1]^d}|x-y| = 2R\sqrt{d}.$$

Since the packing ball radius is $r_t/(2\sqrt{d})$ in our case, it follows that

$$\text{for all }\ x\in B(0,r_0),\quad \min_{1\le j\le n_t}\max_{z\in\widehat{B}_j}|x-z| < r_t.$$

For $1\le j\le n_t$, let $x_j = y_j+\theta\sqrt{2\beta}t\mathbf{e}$, and set as before $B_t^j = B(x_j, r_t/(2\sqrt{d}))$. Then, by translation invariance,

$$\text{for all }\ x\in B_t,\quad \min_{1\le j\le n_t}\max_{z\in B_t^j}|x-z| < r_t. \tag{5.19}$$

(We take the distance between a point in space and the farthest point of the packing ball that is closest to it to cover the 'worst' case, corresponding to Z hitting the farthest point of the closest B_t^j.) Define $\widehat{A}_t := \{\mathrm{supp}(Z(t))\text{ is not } r_t\text{-dense in } B_t\}$. Then, with the choice of the collection $(x_j : 1\le j\le n_t)$, the event A_t from part I of the proof satisfies $\widehat{A}_t\subseteq A_t$. Indeed, if $(A_t)^c$ occurs, then by the definition of the event A_t, $Z_t(B_t^j)>0$ for each $1 \le j \le n_t$, which, by (5.19), implies that each point in B_t has a particle of $Z(t)$ within distance less than r_t. Then, (5.18) implies that

$$\limsup_{t\to\infty} \frac{1}{t} \log P\left(\widehat{A}_t\right) \le -\beta I(\theta, k, 0). \tag{5.20}$$

5.2.3. Part III: Extension from B_t to the entire subcritical ball. By a geometric argument similar to the one in part II of this proof, we extend the result on the density of BBM in B_t to the density in the entire subcritical ball $\mathcal{B}_t$. Recall that $\rho_t = \theta\sqrt{2\beta}t$, and define

$$m_t := 2^d \left\lceil \sqrt{d}\, \rho_t \frac{1}{r_0} \right\rceil^d.$$

Let $(\bar{x}_j : 1 \le j \le m_t)$ be any collection of m_t points in $\mathcal{B}_t := B(0, \rho_t)$. For each j, define $\mathcal{B}_t^j := B(\bar{x}_j, r_0)$. Next, for $t > 0$ and $1 \le j \le m_t$, define the events

$$E_t^j := \{\text{supp}(Z(t)) \text{ is not } r_t\text{-dense in } \mathcal{B}_t^j\}, \quad E_t := \bigcup_{1\le j\le m_t} E_t^j.$$

Recall that $B_t = B(\rho_t \mathbf{e}, r_0)$, and $|\bar{x}_j| \le \rho_t$ for all j. Then, since the function $I = I(\theta, k, a)$ is decreasing in parameter θ when the other two parameters are fixed, it follows from (5.20) that for each $j \in \{1, \dots, m_t\}$,

$$\limsup_{t\to\infty} \frac{1}{t} \log P\left(E_t^j\right) \le -\beta I(\theta, k, 0).$$

Then, using the union bound and that m_t is only a polynomial factor, we obtain

$$\limsup_{t\to\infty} \frac{1}{t} \log P\left(E_t\right) \le \limsup_{t\to\infty} \frac{1}{t} \log \left[m_t \max_{1\le j\le m_t} P\left(E_t^j\right)\right] \le -\beta I(\theta, k, 0). \tag{5.21}$$

We now choose the collection $(\bar{x}_j : 1 \le j \le m_t)$ in a useful way. Let $C(0, \rho_t)$ be the cube centered at the origin with side length $2\rho_t$. The simple cubic packing of $C(0, \rho_t)$ requires at most m_t balls of radius $r_0/(2\sqrt{d})$, say with centers $(\bar{y}_j : 1 \le j \le m_t)$. By an argument similar to the one in part II of this proof, one can show that

$$\forall\, x \in B(0, \rho_t), \quad \min_{1\le j\le m_t} |x - \bar{x}_j| < r_0,$$

which implies that

$$B(0, \rho_t) \subseteq \bigcup_{1\le j\le m_t} B(\bar{x}_j, r_0).$$

Here, we are enlarging the packing ball radius from $r_0/(2\sqrt{d})$ to r_0 so that every point in $B(0, \rho_t)$ falls inside at least one enlarged ball $B(\bar{x}_j, r_0)$. Then, with the choice of the collection $(\bar{x}_j : 1 \le j \le m_t)$, the event E_t from above satisfies

$$A_t^r = \{\text{supp}(Z(t)) \text{ is not } r_t\text{-dense in } B(0, \rho_t)\} \subseteq \bigcup_{1\le j\le m_t} E_t^j = E_t,$$

and (5.21) implies that

$$\limsup_{t\to\infty} \frac{1}{t} \log P\left(A_t^r\right) \le -\beta I(\theta, k, 0).$$

This completes the proof of the upper bound of Theorem 2.4.

6. Enlargement of BBM

For a BBM $Z = (Z(t))_{t\ge 0}$, recall the definition of its *r-enlargement at time t* as

$$Z_t^r := \bigcup_{x \in \text{supp}(Z(t))} B(x, r).$$

6.1. Proof of Theorem 2.8

We will show that for every $\varepsilon > 0$ there exist $c_1 > 0$ and $c_2 > 0$ such that for all large t,

$$P\left(\frac{\mathsf{vol}\,(Z_t^{r_t})}{t^d} \leq [2\beta(1-kd-\varepsilon)]^{d/2}\omega_d\right) \leq e^{-c_1 t}, \tag{6.1}$$

and

$$P\left(\frac{\mathsf{vol}\,(Z_t^{r_t})}{t^d} \geq [2\beta(1-kd+\varepsilon)]^{d/2}\omega_d\right) \leq e^{-c_2 t}. \tag{6.2}$$

Let $\varepsilon > 0$ and set $\theta = \theta_1 = \sqrt{1-kd-\varepsilon/2}$ in Theorem 2.4, which gives $\rho_t = \theta_1\sqrt{2\beta}t = \sqrt{2\beta(1-kd-\varepsilon/2)}t$. Then, $0 \leq k < (1-\theta_1^2)/d = k+\varepsilon/(2d)$ so that Theorem 2.4 applies, and gives

$$P(A_t^r) = \exp[-\beta\, I(\theta_1, k, 0)t + o(t)].$$

This proves (6.1) since for all large t, $\left\{\mathsf{vol}\,(Z_t^{r_t})\,/t^d \leq [2\beta(1-kd-\varepsilon)]^{d/2}\omega_d\right\} \subseteq A_t^r$. Indeed, if $(A_t^r)^c = \{\text{supp}(Z(t)) \text{ is } r_t\text{-dense in } B(0,\rho_t)\}$ occurs, then $\bigcup_{x\in\text{supp}(Z(t))} B(x,r_t) \supseteq B(0,\rho_t)$, which implies that $\mathsf{vol}(Z_t^{r_t}) \geq \mathsf{vol}(B(0,\rho_t)) = [2\beta(1-kd-\varepsilon/2)]^{d/2}t^d\omega_d$.

To prove (6.2), for $\theta \geq 0$ and $t \geq 0$, let $\mathcal{N}_t^\theta$ be the set of particles outside $B(0,\theta\sqrt{2\beta}t)$ at time t. Set $\theta_2 = \sqrt{1-kd+\varepsilon/2}$. Then,

$$E(|\mathcal{N}_t^{\theta_2}|) = \exp[\beta t(1-\theta_2^2)+o(t)] = \exp[\beta t(kd-\varepsilon/2)+o(t)],$$

and the Markov inequality yields

$$P\left(|\mathcal{N}_t^{\theta_2}| \geq \exp[\beta t(kd-\varepsilon/4)+o(t)]\right) \leq \exp[-\beta t\varepsilon/4+o(t)]. \tag{6.3}$$

For an upper bound on $\mathsf{vol}\,(Z_t^{r_t})$, suppose that $B(0,\theta_2\sqrt{2\beta}t) \subseteq Z_t^{r_t}$ and that the balls of radius r_t centered at the positions of the particles at time t are all disjoint from one another. Since the volume of a ball of radius r_t is $\omega_d(r_0e^{-\beta kt})^d$, by choosing ε with $\varepsilon/4 < k$, it then follows from (6.3) that

$$\begin{aligned}
&P\left(\mathsf{vol}\left(Z_t^{r_t}\cap(B(0,\theta_2\sqrt{2\beta}t))^c\right) \geq \omega_d r_0^d\exp[-\beta t\varepsilon/4]\right)\\
&\quad\leq P\left(|\mathcal{N}_t^{\theta_2}|\mathsf{vol}(B(0,r_t)) + \omega_d\left[(r_t+\theta_2\sqrt{2\beta}t)^d - (\theta_2\sqrt{2\beta}t)^d\right] \geq \omega_d r_0^d\exp[-\beta t\varepsilon/4]\right)\\
&\quad= P\left(|\mathcal{N}_t^{\theta_2}| \geq \exp[\beta t(kd-\varepsilon/4)+o(t)]\right)\\
&\quad\leq \exp[-\beta t\varepsilon/4+o(t)],
\end{aligned}$$

where we used that $(r_t+\theta_2\sqrt{2\beta}t)^d - (\theta_2\sqrt{2\beta}t)^d \leq r_te^{o(t)}$, and used A^c to denote the complement of a set A in $\mathbb{R}^d$. This implies (6.2), and completes the proof of Theorem 2.8.

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